Vertex operator algebras with central charges 1/2 and −68/7 (v.5)

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Lie algebras, Vertex Operator Algebras, and Related Topics
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Introduction

1. Vertex operator algebras (VOA for short) with central charges $1/2$ or $-68/7$ whose sets of characters form fundamental systems of 3rd order modular linear differential equations (MLDE for short) are discussed.
   - Such VOAs are either isomorphic to the minimal series of Virasoro VOAs with central charges $c = c_{3,4} = 1/2$ or $c_{2,7} = -68/7$.
2. We also study VOAs with central charges 8 or 16.
   - The lattice vertex operator algebras V_L , where L is the $\sqrt{2}E_8$ ($c = 8$) or the Barnes–Wall lattice ($c = 16$) appear.
3. The characters of each VOA are obtained by **solving** the associated with MLDE (by Frobenius method).

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3rd oder modular linear differential equations

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$$D^3(f)-\frac{1}{2}E_2D(f)+\left\{\frac{1}{2}E_2'-\left(h^2-\frac{h}{2}-\frac{ch}{8}+\frac{c^2}{192}+\frac{c}{24}\right)E_4\right\}D(f)$$
$$+\frac{c}{24}\left(\frac{c}{12}+\frac{1}{2}-h\right)\left(h-\frac{c}{24}\right)E_6 f = 0, \quad D = {}' = q\frac{d}{dq}.$$

2. The c is a central charge and h is the minimal conformal weight and $E_k(q)$ is the Eisenstein series with weight k.
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Frobenius method

1. A solution of the form $f = \sum_{n=0}^{\infty} a_n q^{\varepsilon+n}$ with $a_0 = 1$ and $\varepsilon \in \mathbb{Q}$. We suppose that an index is a rational number.
2. The index $\varepsilon \in \{-c/24,\ h - c/24,\ c/12 - h + 1/2\}$.
3. The *Frobenius method* determines a_n $(n \in \mathbb{N})$ uniquely by the recursion relation for a given $a_0 \neq 0$:

$$a_n = \left(n+\varepsilon+\frac{c}{24}\right)^{-1}\left(n+\varepsilon+\frac{c}{24}-h\right)^{-1}\left(n+\varepsilon-\frac{c}{12}-\frac{1}{2}+h\right)^{-1}$$
$$\times\sum_{i=1}^{n}\{(\varepsilon-i+n)(12(2i-\varepsilon-n)\sigma_1(i)$$
$$+\frac{5}{4}\left(c^2+8c-24hc-96h+192h^2\right)\sigma_3(i))$$
$$-\frac{7}{96}c(c-24h)(c-12h+6)\sigma_5(i)\}a_{n-i},\ (E_k = 1 - A_k\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n)$$

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The MLDE and conformal weights for $c = 1/2$

1. The MLDE for $c = 1/2$

$$D^3(f) - \frac{1}{2}E_2D(f) + \left\{\frac{1}{2}E_2' - \left(17/768 - \frac{9h}{16} + h^2\right)E_4\right\}D(f) + \frac{1}{48}\left(\frac{13}{24} - h\right)\left(-\frac{1}{48} + h\right)E_6 f = 0,$$

2. Let f_0 be the formal solution (q-series) with the index $(-c/24 =) - 1/48$. Then the second coefficient of f_0 is

$$m := a_1 = \frac{31\,(32h^2 - 18h + 1)}{4(h-1)(16h+7)}.$$

3. m is an integer. $\Longleftrightarrow$ The quadratic equation in h must have an integral square discriminant d^2 ($d \in \mathbb{Z}$) since h is rational. $\Longleftrightarrow$

$(1058m - 23d - 8959)(1058m + 23d - 8959) = 55353600.$

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Conformal weights for $c = 1/2$ (1)

1. All solutions (m, d) are given by

$$\left\{(-166, \pm 8019), (0, \pm 217), (23, \pm 585), (93, \pm 3875)\right\}.$$

2. The set of h with integral a_1 are

$$\left\{-\frac{15}{16}, -\frac{1}{2}, -\frac{9}{22}, \frac{1}{16}, \frac{1}{2}, \frac{171}{176}, \frac{17}{16}, \frac{3}{2}\right\}.$$

3. Some pairs of values of h give the same set of conformal weights (and then indices).

Values of h (Conformal weights)	Indices
$171/176, -9/22$	$-1/48, 251/264, -227/528$
$1/2, 1/16$	$-1/48, 23/48, 1/24$
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2. The several terms of q-expansions of f_h (whose conformal weights are h) of the index $-1/48$.

$$f_{\frac{171}{176}} = \frac{1}{\sqrt[48]{q}} - 166q^{47/48} + \cdots, \quad f_{\frac{1}{2}} = \frac{1}{\sqrt[48]{q}} + q^{95/48} + q^{143/48} + \cdots,$$

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3. $h \neq 171/176$ ($a_1 < 0$), $h \neq -1/2$ ($a_3 \notin \mathbb{Z}$). $h \neq -15/16$ since the solution with central charge 3/2 has negative coefficients ($a_6 = -31299$).
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Theorem ($c = 1/2$)

Theorem 1

Let V be a vertex operator algebra with central charge $1/2$. Suppose that

(a) *The conformal weights are rational numbers,*
(b) *The space of characters is 3-dimensional,*
(c) *The set of characters forms a fundamental system of solutions of a 3rd order MLDE.*

Then V is isomorphic to the Virasoro vertex operator algebra with central charge $1/2$ and conformal weight $\{0, 1/2, 1/16\}$.

Proof

Proof.

1. The uniqueness of solutions of the MLDE shows $\mathcal{X}_V = \mathcal{X}_{V_{1/2}}$.
2. Let V^{ω} be the vertex operator subalgebra generated by the Virasoro element $\omega \in V_2$.
3. $\mathrm{ch}_{V^{\omega}} \leq \mathrm{ch}_V$.
4. Then either $V^{\omega} \cong M(1/2,0)/\langle L_{-1}\mathbf{1}\rangle$ or $V_{1/2}$.
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Corollary

(d) Let 0, h_1 and h_2 be conformal weights. Then $h_1 + h_2 = 13/24$.

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The condition (c) *is replaced by* (d).

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1. By (d) (formal) characters are formal solutions of a MLDE.
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The characters for $c = 1/2$

1. The character of the VOA $V_{1/2}$

$$\mathrm{ch}_{V_{1/2}}(\tau) = \frac{\phi_1(q) + \phi_2(q)}{2} = \eta(q^2)^{-1} \sum_{n \in \mathbb{Z}} q^{(2n+1/4)^2}.$$

2. The character of the $V_{1/2}$-module $V_{1/2}(1/2)$

$$\mathrm{ch}_{1/2} = \frac{\phi_1(q) - \phi_2(q)}{2} = \eta(q^2)^{-1} \sum_{n \in \mathbb{Z}} q^{(2n+3/4)^2}.$$

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1. Indeed, characters of the Ising model are known.
2. In this talk every character is obtained by using the Frobenius method and the theory of modular forms (with helps of computer and "The On-Line Encyclopedia of Integer Sequences® (OEIS®)" (https://oeis.org/?language=english)
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Vertex operator algebras with central charge $-68/7$
Frobenius method for $c=-68/7$

1. Let $f_0=\sum_{j=0}^{\infty} b_j q^{17/42+j}$ with $b_0=1$.
2. The second coefficient b_1 is given by
$$b_1 = -\frac{2108\left(49h^2+35h+6\right)}{49(h-1)(7h+12)}. \qquad (1)$$
3. Eq. (1) is rewritten as ($m=b_1\in\mathbb{Z}_{\geq 0}$)
$$(103292-343m)h^2+(73780-245m)h + 588m+12648 = 0. \qquad (2)$$
4. m is an integer and h is a rational number. $\Longleftrightarrow$
The discriminant of (1) is d^2 for some $d\in\mathbb{Z}$. $\Longleftrightarrow$
$$81(31-2m)^2-32(31-2m)(28m+31) = d^2.$$

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Conformal weights for $c=-68/7$

1. The list of values of h (Step 1).
   $132/133$, $-227/133$, $72/77$, $-127/77$, $179/203$, $-324/203$, $5/7$, $-10/7$, $-8/77$, $-47/77$ $-2/7$, $-3/7$. $(\sharp=12)$
2. $h\neq -227/133, -127/77, -324/203, -10/7 \Leftarrow$ The b_1 of the q-series with the index $h+59/42$ are negative.
3. The list of values of h (Step 2).
   $132/133$, $72/77$, $179/203$, $5/7$, $-8/77$, $-47/77$, $-2/7$, $-3/7$. $(\sharp=8)$
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6. $h\neq -8/77, -47/77. \Leftarrow$ The b_2 of the q-series with the index $17/42$ are not integers.
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Theorem ($c = -68/7$)

Theorem 3

Let V be a vertex operator algebra (of CFT type) with central charge $-68/7$. Suppose that

(a) *The conformal weights are rational numbers,*
(b) *The space of characters is 3-dimensional,*
(c) *The set of characters forms a fundamental system of solutions of a 3rd order MLDE.*

Then V is isomorphic to the Virasoro vertex operator algebra with central charge $-68/7$ and conformal weight $\{0, -2/7, -3/7\}$.

Proof.

Use the same discussion given for $c = 1/2$. □

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MLDE ($c = -68/7$)

The MLDE

$$D^3(f) - \frac{1}{2}E_2 D^2(f) + \left(\frac{1}{2}E_2' + \frac{1}{28}E_4\right) f' + \frac{85}{74088}E_6 f = 0 .$$

The characters ($c = -68/7$)

1. The characters are given by

$$\begin{aligned}\mathrm{ch}_V(\tau) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(-1)^n q^{(14n+5)^2/56} \\ &= q^{17/42} \prod_{\substack{n>0 \\ n\not\equiv 0,\pm 1 \bmod 7}} (1-q^n)^{-1},\end{aligned}$$

$$\begin{aligned}\mathrm{ch}_{-2/7}(\tau) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(-1)^n q^{(14n+3)^2/56} \\ &= q^{5/42} \prod_{\substack{n>0 \\ n\not\equiv 0,\pm 2 \bmod 7}} (1-q^n)^{-1},\end{aligned}$$

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Vertex operator algebras with central charge $c = 8$

Definition 4

Let V and W be vertex operator algebras. If V and W has the same space of characters we say that V and W are **pseudo-isomorphic**.

1. We study VOAs whose central charge is 8.
2. A typical example is V_L where $L = \sqrt{2}E_8$.

Theorem 5

Let V be a vertex operator algebra with central charge 8 and the space of solutions which gives a fundamental system of solutions of a 3rd order MLDE. Then V is pseudo-isomorphic to V_L where $L = \sqrt{2}E_8$.

Vertex operator algebras with central charge $c = 8$

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Vertex operator algebras with central charge $c = 16$

Theorem 6

Let V be a vertex operator algebra with central charge 16, rational conformal weights, and the space of solutions which gives a fundamental system of solutions of a 3rd order MLDE. Then V is pseudo-isomorphic to either the Barnes-Wall lattice $(c = 16, h = 1)$ vertex operator algebra, the affine VOA of type D_{16} $(c = 16, h = 2)$ and level 1 and the affine VOA of type D_{28} with level 1 $(c = 28, h = 3)$.

Vertex operator algebras with central charge $c = 16$

Theorem 7

Let V be a vertex operator algebra with $c = 16$ and $h = 1$. Then the conformal weighs is $\{0, 1, 3/2\}$ and V is pseudo-isomorphic to the Barnes-Wall lattice vertex operator algebra V_L. The set of characters is given by

$$\mathrm{ch}_V(\tau) = x(q)^4 - 96x(q)^2y(q)^2 + 6144y(q)^4\,,$$
$$\mathrm{ch}_1(\tau) = 32y(q)^2(x(q)^2 + 64y(q)^2)\,,$$
$$\mathrm{ch}_{3/2}(\tau) = 512x(q)y(q)^3\,.$$

Further V is pseudo-isomorphic to the orbifold V_L^+ whose sets of conformal weights and characters are same as those of V.

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Modular forms

$$x(q) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} (-1)^n q^{(18n+7)^2/72},$$

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Vertex operator algebras with central charge $c = 16$

Theorem 8

Let V be a vertex operator algebra with $c = 16$ and $h = 2$. Then the conformal weights is $\{0, 1/2, 2\}$ and V is pseudo-isomorphic to the affine VOA of type D_{16} and level 1. The set of characters is given by

$$\mathrm{ch}_V = (\phi_1(q)^{32} + \phi_2(q)^{32})/2\,, \quad \mathrm{ch}_{1/2} = (\phi_1(q)^{32} - \phi_2(q)^{32})/2\,,$$

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$$\mathrm{ch}_V(\tau) = \frac{\phi_1(q)^{56} + \phi_2(q)^{56}}{2},$$

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Answer to expected questions

1. What happens when the minimal model has 4 simple modules?
Answer. We cannot characterize these minimal models by their central charges. One candidate of conditions is

$$\mathrm{ch}_V = q^{-c/24}\left(1 + 0\cdot q + mq^2 + \cdots\right), \quad (m \in \mathbb{N}).$$

We are working on up to 6.

2. Can you characterize the whole minimal models?
Answer. It promises. Probably we have to use the result of C. Dong and W. Zhang (JA.)

3. Can you classify affine VOAs by their central charges?
Answer. Basically, yes! A condition for this is that

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4. Can you classify lattice VOAs by their central charges?
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