Holomorphic symplectic fermions

Ingo Runkel
Hamburg University

joint with Alexei Davydov

Outline

Investigate holomorphic extensions of symplectic fermions

- via embedding into a holomorphic VOA
  (existence)
- via study of commutative algebras (+ extra properties) in a braided tensor category
  (uniqueness – up to two assumptions)

Why?

Symplectic fermions:

- first described in [Kausch '95], by now best studied example of a logarithmic CFT
- L_0-action on representations may not be diagonalisable, thus have non-semisimple representation theory
- finite number of irreducible representations, projective covers have finite length, ...

Holomorphic VOAs:

- VOAs $\mathbb{V}$, all of whose modules are isom. to direct sums of $\mathbb{V}$
- have (almost) modular invariant character
- all examples I know are lattice VOAs and orbifolds thereof

Can one find new examples by studying extensions of VOAs which have logarithmic modules?

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Can one find new examples by studying extensions of VOAs which have logarithmic modules?

Answer for symplectic fermions (up to two assumptions) : No.

Free super-bosons

Following [Frenkel, Lepowsky, Meurmann '87], [Kac '98] (see [IR '12] for treatment of non-semisimple aspects):

Fix

- a finite-dimensional super-vector space $\mathfrak{h}$
- a super-symmetric non-degenerate pairing $(-,-) : \mathfrak{h} \otimes \mathfrak{h} \to \mathbb{C}$

  i.e. $(a,b) = \delta_{|a|,|b|}(-1)^{|a|}(b,a)$

Define the affine Lie super-algebras

$$\begin{aligned} \widehat{\mathfrak{h}} &= \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \\ \widehat{\mathfrak{h}}_{\mathrm{tw}} &= \mathfrak{h} \otimes_{\mathbb{C}} t^{\frac{1}{2}}\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \end{aligned}$$

with K even and central and super-bracket

$$[a_m, b_n] = (a,b)\, m\, \delta_{m+n,0}\, K$$

$a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$ resp. $m, n \in \mathbb{Z} + \frac{1}{2}$.

. . . free super-bosons – representations

Consider $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}_{\mathrm{tw}}$ modules M

- where K acts as id
- which are *bounded below*: For each $v \in M$ there is $N > 0$ s.t. $a^1_{m_1} \cdots a^k_{m_k} v = 0$ for all $a^i \in \mathfrak{h}$, $m_1 + \cdots + m_k \geq N$.
- where the *space of ground states*

$$M_{\mathrm{gnd}} = \{ v \in M \,|\, a_m v = 0 \text{ for all } a \in \mathfrak{h}, m > 0 \}$$

  is finite-dimensional.

Categories of representations of above type:

$$\mathrm{Rep}^{\mathrm{fd}}_{\flat,1}(\widehat{\mathfrak{h}}) \qquad \text{and} \qquad \mathrm{Rep}^{\mathrm{fd}}_{\flat,1}(\widehat{\mathfrak{h}}_{\mathrm{tw}})$$

. . . free super-bosons – representations

$\mathrm{Rep}^{\mathrm{fd}}(\mathfrak{h})$: category of finite-dimensional $\mathfrak{h}$-modules
(not semisimple)

Induction:

- $N \in \mathrm{Rep}^{\mathrm{fd}}(\mathfrak{h})$ gives $\widehat{\mathfrak{h}}$-module

$$\widehat{N} := U(\widehat{\mathfrak{h}}) \otimes_{U(\mathfrak{h})_{\geq 0} \oplus \mathbb{C}K} N$$

(K acts as 1, a_m with $m > 0$ acts as zero, a_0 acts as a)

- super-vector space V gives $\widehat{\mathfrak{h}}_{\mathrm{tw}}$-module

$$\widehat{N} := U(\widehat{\mathfrak{h}}_{\mathrm{tw}}) \otimes_{U(\mathfrak{h}_{\mathrm{tw}})_{>0} \oplus \mathbb{C}K} V$$

Theorem: The following functors are mutually inverse equivalences:

$$\mathrm{Rep}^{\mathrm{fd}}_{\flat,1}(\widehat{\mathfrak{h}}) \underset{\widehat{(-)}}{\overset{(-)_{\mathrm{gnd}}}{\rightleftarrows}} \mathrm{Rep}^{\mathrm{fd}}(\mathfrak{h}) \quad \text{and} \quad \mathrm{Rep}^{\mathrm{fd}}_{\flat,1}(\widehat{\mathfrak{h}}_{\mathrm{tw}}) \underset{\widehat{(-)}}{\overset{(-)_{\mathrm{gnd}}}{\rightleftarrows}} s\mathcal{V}ect^{\mathrm{fd}}$$

Symplectic fermions

The vacuum $\widehat{\mathfrak{h}}$-module $\widehat{\mathbb{C}^{1|0}}$ is a vertex operator super-algebra (VOSA) (central charge is super-dimension of $\mathfrak{h}$) .

For $\mathfrak{h}$ purely odd, this is the *symplectic fermion VOSA* $\mathbb{V}(d)$, where $\dim \mathfrak{h} = d$.

$\mathbb{V}(d)_{\mathrm{ev}}$: parity-even subspace, a vertex operator algebra (VOA).

Properties of $\mathbb{V}(d)_{\mathrm{ev}}$: Abe '05

- central charge $c = -d$
- C_2-cofinite
- has 4 irreducible representations

	S_1	S_2	S_3	S_4
lowest L_0-weight	0	1	$-\frac{d}{16}$	$-\frac{d}{16} + \frac{1}{2}$
character	$\chi_1(\tau)$	$\chi_2(\tau)$	$\chi_3(\tau)$	$\chi_4(\tau)$

- $\mathrm{Rep}\mathbb{V}(d)_{\mathrm{ev}}$ is not semisimple

Modular invariance

Question:

Are there non-zero linear combinations of $\chi_1, \ldots, \chi_4$ with non-negative integral coefficients which are "almost" modular invariant?

Almost modular invariant:
$f(-1/\tau) = \xi\, f(\tau)$ and $f(\tau+1) = \zeta\, f(\tau)$ for some $\xi, \zeta \in \mathbb{C}^\times$

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Answer: Davydov, IR ≈'15

Combinations as above exist iff $d \in 16\mathbb{Z}_{>0}$. In this case, the only possibilities are $Z(\tau) = z_1\chi_1(\tau) + \cdots + z_4\chi_4(\tau)$ with

$$(z_1, z_2, z_3, z_4) \in (2^{\frac{d}{2}-1}, 2^{\frac{d}{2}-1}, 1, 0)\,\mathbb{Z}_{>0} \ .$$

$Z(\tau)$ is modular invariant iff $d \in 48\mathbb{Z}_{>0}$.

More questions

Minimal almost modular invariant solution:

$$Z_{\min}(\tau) = 2^{\frac{d}{2}-1}\big(\chi_1(\tau) + \chi_2(\tau)\big) + \chi_3(\tau) \qquad (d \in 16\mathbb{Z}_{>0})$$

Questions:

Q1 Is $Z_{\min}(\tau)$ the character of a holomorphic extension of $\mathbb{V}(d)_{\mathrm{ev}}$?

Q2 What are all holomorphic extensions of $\mathbb{V}(d)_{\mathrm{ev}}$?

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$$= \frac{1}{2}\,\eta(\tau)^{-\frac{d}{2}}\Big(\theta_2(\tau)^{\frac{d}{2}} + \theta_3(\tau)^{\frac{d}{2}} + \theta_4(\tau)^{\frac{d}{2}}\Big)$$

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Q1 Is $Z_{\min}(\tau)$ the character of a holomorphic extension of $\mathbb{V}(d)_{\mathrm{ev}}$?

Q2 What are all holomorphic extensions of $\mathbb{V}(d)_{\mathrm{ev}}$?

Summary of the answers:

Q1 Yes, a possible extension is the lattice VOA for the even self-dual lattice $D^+_{d/2}$ with modified stress tensor.

Q2 Making two assumptions, the answer to Q1 provides all holomorphic extensions of $\mathbb{V}(d)_{\mathrm{ev}}$.

Symplectic fermions as a sub-VOSA

Lattice $\mathbb{Z}^r$ with standard inner product gives VOSA $\mathbb{V}_{\mathbb{Z}^r}$.

Sublattice $D_r = \{m \in \mathbb{Z}^r \,|\, m_1 + \cdots + m_r \in 2\mathbb{Z}\}$ is $so(2r)$ root lattice and gives parity-even part:

$$(\mathbb{V}_{\mathbb{Z}^r})_{\mathrm{ev}} = \mathbb{V}_{D_r} \ .$$

We will need a non-standard stress tensor (aka. conformal vector or Virasoro element) for $\mathbb{V}_{\mathbb{Z}^r}$ given by

$$T^{FF} = \frac{1}{2}\sum_{i=1}^{r}(H^i_{-1}H^i_{-1} - H^i_{-2})\Omega \ ,$$

where $H^i_m, i = 1, \ldots, r$ generate Heisenberg algebra $\mathrm{Hei}(r)$.
Central charge $c^{FF} = -2r$.

Appear e.g. as “free boson with background charge” or “Feigin-Fuchs free boson”. Detailed study in context of lattice VOAs in [Dong, Mason ’04].

…symplectic fermions as a sub-VOSA

Theorem: Davydov, IR ≈'15

For every $r \in \mathbb{Z}_{>0}$, there is an embedding $\iota : \mathbb{V}(2r) \to \mathbb{V}_{\mathbb{Z}^r}$ of VOSAs which satisfies $\iota(T^{SF}) = T^{FF}$.

Sketch of proof:

- Pick a symplectic basis $a^1, \dots, a^r, b^1, \dots, b^r$ of $\mathfrak{h}$, s.t. $(a^i, b^j) = \delta_{i,j}$.
- $\mathbb{V}(2r)$ generated by $a^i(x), b^i(x)$, OPE

$$a^i(x) b^j(0) = \delta_{i,j}\, x^{-2} + \text{reg}$$

- Free field construction: Kausch '95, Fuchs, Hwang, Semikhatov, Tipunin '03

  For $m \in \mathbb{Z}^r$ write $|m\rangle$ for corresponding highest weight state in $\mathbb{V}_{\mathbb{Z}^r}$. Then (e_i: standard basis vectors of $\mathbb{Z}^r$)

$$f^{*i} := |e_i\rangle \qquad \text{and} \qquad f^i := -H^i_{-1}\,|{-e_i}\rangle$$

  have OPE in $f^{*i}(x) f^j(0) = \delta_{i,j}\, x^{-2} + \text{reg}$.
- $\widehat{\mathfrak{h}}$-module generated by $\Omega \in \mathbb{V}_{\mathbb{Z}^r}$ is isomorphic to $\mathbb{V}(2r)$.

...symplectic fermions as a sub-VOSA

For $r \in 8\mathbb{Z}$ have the even self-dual lattice

$$D_r^+ = D_r \cup (D_r + [1])$$

with $[1] = (\frac{1}{2}, \dots, \frac{1}{2})$.

In particular, $\mathbb{V}_{D_r} \subset \mathbb{V}_{D_r^+}$. Since $(\mathbb{V}_{\mathbb{Z}^r})_{\mathrm{ev}} = \mathbb{V}_{D_r}$ get:

Corollary:
$\mathbb{V}(2r)_{\mathrm{ev}}$ is a sub-VOA of $\mathbb{V}_{D_r^+}$.

Recall questions:

Q1 Is $Z_{\min}(\tau)$ the character of a holomorphic extension of $\mathbb{V}(d)_{\mathrm{ev}}$?

Q2 What are all holomorphic extensions of $\mathbb{V}(d)_{\mathrm{ev}}$?

Summary of the answers:

Q1 Yes, a possible extension is the lattice VOA for the even self-dual lattice $D_{d/2}^{+}$ with modified stress tensor. ✓

Q2 Making two assumptions, the answer to Q1 provides all holomorphic extensions of $\mathbb{V}(d)_{\mathrm{ev}}$. – next

The braided tensor category $SF(d)$

Take super-vector space $\mathfrak{h}$ to be purely odd.

Had equivalences

$$\mathrm{Rep}^{\mathrm{fd}}_{\flat,1}(\widehat{\mathfrak{h}}) \cong \mathrm{Rep}^{\mathrm{fd}}(\mathfrak{h}) \quad , \quad \mathrm{Rep}^{\mathrm{fd}}_{\flat,1}(\widehat{\mathfrak{h}}_{\mathrm{tw}}) \cong s\mathcal{V}ect^{\mathrm{fd}}$$

Write $SF(d) = SF_0 + SF_1$ with

$$SF_0 = \mathrm{Rep}^{\mathrm{fd}}(\mathfrak{h}) \quad , \quad SF_1 = s\mathcal{V}ect^{\mathrm{fd}} \ .$$

Aim:

1. Use "vertex operators" and conformal blocks for $\widehat{\mathfrak{h}}_{(\mathrm{tw})}$ to equip $SF(d)$ with
   - tensor product
   - associator
   - braiding
2. Find commutative algebras in $SF(d)$ with certain extra properties

Vertex operators

A slight generalisation of free boson vertex operators: IR '12

Definition:

Let $A, B, C \in SF_0$. A *vertex operator from* A, B *to* C is a map

$$V \;:\; \mathbb{R}_{>0} \times (A \otimes \widehat{B}) \longrightarrow \overline{\widehat{C}}$$

($\overline{\widehat{C}}$ is the algebraic completion) such that

(i) even linear in $A \otimes \widehat{B}$, smooth in x

(ii) $L_{-1} \circ V(x) - V(x) \circ (id_A \otimes L_{-1}) = \frac{d}{dx} V(x)$

(iii) for all $a \in \mathfrak{h}$, $a_m \, V(x) = V(x)\big(x^m a \otimes id + id \otimes a_m\big)$

$+$ three more version when any two of A, B, C are in SF_1 .

Vector space of all vertex operators from A, B to C :

$$\mathcal{V}^{C}_{A,B}$$

Same definition works for super-vector spaces $\mathfrak{h}$ which are not purely odd,
i.e. for free super-bosons in general.

Tensor product

Definition:
The tensor product $A * B$ of $A, B \in SF(d)$ is a representing object for the functor $C \mapsto \mathcal{V}_{A,B}^{C}$.

That is, there are isomorphism, natural in C ,

$$\mathcal{V}_{A,B}^{C} \longrightarrow SF(A * B, C) \ .$$

Write $P_{\mathrm{gnd}} : \widehat{A} \to A$ for the projector to ground states.

Theorem: IR '12
The map $V \mapsto P_{\mathrm{gnd}} \circ V(1)$,

$$\mathcal{V}_{A,B}^{C} \to \begin{cases} \mathrm{Hom}_{\mathfrak{h}}(A \otimes B, C) & ; \ A, B, C \in SF_0 \\ \mathrm{Hom}_{s\mathcal{V}ect}(A \otimes B, C) & ; \ \text{else} \end{cases}$$

is an isomorphism, natural in A, B, C .

... tensor product

Recall: $SF(d) = SF_0 + SF_1$, $SF_0 = \mathrm{Rep}^{\mathrm{fd}}(\mathfrak{h})$, $SF_1 = s\mathcal{V}ect^{\mathrm{fd}}$

Combine results:

A	B	$\mathcal{V}^C_{A,B} \cong$	need to find $\cong$ to	$A * B$
0	0	$\mathrm{Hom}_{\mathfrak{h}}(A \otimes B, C)$	$\mathrm{Hom}_{\mathfrak{h}}(A * B, C)$	
0	1	$\mathrm{Hom}_{s\mathcal{V}ect}(A \otimes B, C)$	$\mathrm{Hom}_{s\mathcal{V}ect}(A * B, C)$	
1	0	$\mathrm{Hom}_{s\mathcal{V}ect}(A \otimes B, C)$	$\mathrm{Hom}_{s\mathcal{V}ect}(A * B, C)$	
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Up to here everything worked for general free super-bosons. But to have $U(\mathfrak{h}) \otimes A \otimes B \in SF(d)$, $U(\mathfrak{h})$ must be *finite-dimensional*. Thus need $\mathfrak{h}$ purely odd.

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$SF(d)$ has four simple objects (Π : parity shift):

$$\mathbf{1} = \mathbb{C}^{1|0} \in SF_0 \quad , \quad \Pi\mathbf{1} \quad , \quad T = \mathbb{C}^{1|0} \in SF_1 \quad , \quad \Pi T$$

For example, $T * T = U(\mathfrak{h})$ is reducible but indecomposable.

First noticed in triplet model $W(2) = \mathbb{V}(2)_{\mathrm{ev}}$ in [Gaberdiel, Kausch '96].

...tensor product

Next compute

- braiding from monodromy of conformal 3-point blocks
  (= vertex operators) ,
- associator from asymptotic behaviour of 4-point block
  (computation partly conjectural)

Theorem:
$SF(d)$ is a braided tensor category. In addition, $SF(d)$ can be equipped with duals and a ribbon twist to become a ribbon category.

Relation to $\mathrm{Rep}(\mathbb{V}(d)_{\mathrm{ev}})$

Huang, Lepowsky, Zhang '10-'11

$\mathrm{Rep}(\mathbb{V}(d)_{\mathrm{ev}})$ carries the structure of a braided tensor category.

Conjecture:
The functor $SF(d) \to \mathrm{Rep}(\mathbb{V}(d)_{\mathrm{ev}})$, $A \mapsto (\widehat{A})_{\mathrm{ev}}$ is well-defined and gives an equivalence of braided tensor categories.

object $A \in SF(d)$:	$\mathbf{1}$	$\Pi\mathbf{1}$	T	ΠT
L_0-weight of $(\widehat{A})_{\mathrm{ev}}$:	0	1	$-\frac{d}{16}$	$-\frac{d}{16}+\frac{1}{2}$

Classification of holomorphic extensions

A *holomorphic VOA* is a VOA $\mathbb{V}$ such that all its admissible modules are isomorphic to direct sums of $\mathbb{V}$.

For rational VOAs $\mathbb{V}$ + extra conditions we have

Huang, Kirillov, Lepowsky '15

Theorem:
There is a 1-1 correspondence between holomorphic extensions of $\mathbb{V}$ and Lagrangian algebras in $\mathrm{Rep}(\mathbb{V})$.

Lagrangian algebras

Defined (in modular tensor categories) in [Fröhlich, Fuchs, Schweigert, IR '03] ("trivialising algebra") and [Davydov, Müger, Nikshych, Ostrik '10] ("Lagrangian algebra")

$\mathcal{C}$: braided tensor cat. with duals and ribbon twists (a ribbon category)

Define:

- algebra in $\mathcal{C}$: object $A \in \mathcal{C}$, morphisms $\mu : A \otimes A \to A$, $\eta : \mathbf{1} \to A$, such that associative and unital
- commutative algebra in $\mathcal{C}$: an algebra A such that $\mu \circ c_{A,A} = \mu$ where $c_{U,V} : U \otimes V \to V \otimes U$ is the braiding
- (left A-)module: object $M \in \mathcal{C}$, morphism $\rho : A \otimes M \to M$, such that compatible with μ, η
- local module: module M such that $\rho \circ c_{M,A} \circ c_{A,M} = \rho$

A Lagrangian algebra is a commutative algebra A with trivial twist (i.e. $\theta_A = id_A$), such that every local A-module is isomorphic to a direct sum of A's as a left module over itself.

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Assumption:
This theorem also holds for symplectic fermions, i.e. for $\mathbb{V}(d)_{\mathrm{ev}}$ and $\mathrm{Rep}(\mathbb{V}(d)_{\mathrm{ev}})$

(It should hold for C_2-cofinite VOAs in general.)

...classification of holomorphic extensions

Theorem: Davydov, IR ≈'15

1. For $d \notin 16\mathbb{Z}$, $SF(d)$ contains no Lagrangian algebras whose class in $K_0(SF(d))$ is a multiple of

$$2^{\frac{d}{2}-1}\big([\mathbf{1}] + [\Pi\mathbf{1}] \big) + [T] \,. \qquad (*)$$

2. If $d \in 16\mathbb{Z}$, for each Lagrangian subspace $\mathfrak{f} \subset \mathfrak{h}$, there is a Lagrangian algebra $H(\mathfrak{f}) \in SF(d)$.

   These algebras are mutually non-isomorphic, but any two are related by a braided tensor autoequivalence of $SF(d)$.

3. Any Lagrangian algebra in $SF(d)$ whose class in K_0 is a multiple of $(*)$ is isomorphic to $H(\mathfrak{f})$ for some $\mathfrak{f}$

   (in particular, its class in K_0 is equal to the one in $(*)$) .

...classification of holomorphic extensions

Combine:

- the theorem classifying Lagrangian algebras in $SF(d)$
- the conjecture that $SF(d) \cong \mathrm{Rep}\mathbb{V}(d)_{\mathrm{ev}}$
- the assumption that holomorphic extensions of $\mathbb{V}(d)_{\mathrm{ev}}$ are in 1-1 correspondence to Lagrangian algebras in $\mathrm{Rep}(\mathbb{V}(d)_{\mathrm{ev}})$
- different choices of Lagrangian subspaces $\mathfrak{f} \subset \mathfrak{h}$ in $H(\mathfrak{f})$ lead to isomorphic VOAs (the isomorphism acts non-trivially also on $\mathbb{V}(d)_{\mathrm{ev}}$)

This gives:

For $d \notin 16\mathbb{Z}_{>0}$, $\mathbb{V}(d)_{\mathrm{ev}}$ has no holomorphic extensions.

For $d \in 16\mathbb{Z}_{>0}$, every holomorphic extension of $\mathbb{V}(d)_{\mathrm{ev}}$ is isomorphic to the inclusion $\iota : \mathbb{V}(d)_{\mathrm{ev}} \hookrightarrow \mathbb{V}_{D^+_{d/2}}$ (with stress tensor T^{FF}).