Symmetric polynomials and modules over affine $\mathfrak{sl}(2)$ at admissible levels

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Joint work with David Ridout

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Affine vertex operator algebras

Let $\mathfrak{g}$ be a simple Lie algebra with affinisation $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$.
Then, for $k \neq -h^\vee$,

$$V_k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}_k, \quad K\big|_{\mathbb{C}_k} = k\cdot\mathrm{id}, \quad \mathfrak{g}\otimes\mathbb{C}[t]\big|_{\mathbb{C}_k} = 0,$$

is a *universal affine vertex operator algebra*.

For certain levels k, there exist proper ideals.

$$L_k(\mathfrak{g}) = \frac{V_k(\mathfrak{g})}{\langle \text{max ideal} \rangle}.$$

Idea and goal

Determine module theory of $L_k(\mathfrak{g})$ from that of $V_k(\mathfrak{g})$.

Example $\mathfrak{g} = \mathfrak{sl}(2) = \mathrm{span}\{E, H, F\}$

For $\mathfrak{g} = \mathfrak{sl}(2)$, there exists a (unique) proper ideal I if and only if

$$k+2 = \frac{u}{v}, \quad u \geq 2,\ v \geq 1,\ \gcd(u,v) = 1,$$

$$L_k(\mathfrak{sl}(2)) = \frac{V_k(\mathfrak{sl}(2))}{I}.$$

Such levels are called admissible. The ideal is generated by a singular vector χ of $\mathfrak{sl}(2)$-weight $2(u-1)$ and conformal weight $(u-1)v$.

Integral levels

For $v = 1$,

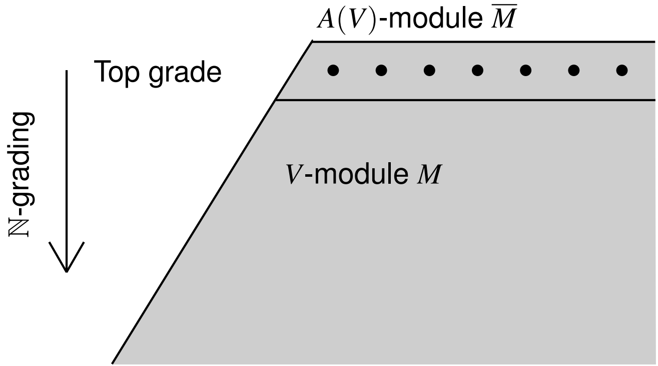
$$\chi = (E_{-1})^{u-1} \mathbf{1}_{u-2}.$$

The Zhu algebra

Moral "definition": Zhu algebra of vertex operator algebra V

$A(V) \simeq \{$0-modes of V acting on vectors annihilated by pos. modes.$\}$

There is a 1-1 correspondence between simple $\mathbb{N}$-gradable modules over a vertex operator algebra V and simple modules over the Zhu algebra $A(V)$.



Classification strategy

Let $\pi : V \twoheadrightarrow A(V)$.

Theorem [Frenkel,Zhu]

- For $V_k(\mathfrak{g})$, Zhu's algebra is $A(V_k(\mathfrak{g})) \simeq U(\mathfrak{g})$.
- For any ideal $I \subset V_k(\mathfrak{g})$, the image $\pi(I)$ is an ideal of $A(V_k(\mathfrak{g}))$ and $A\left(\frac{V_k(\mathfrak{g})}{I}\right) = \frac{A(V_k(\mathfrak{g}))}{\pi(I)}$
- For $\chi \in V_k(\mathfrak{g})$ singular, such that $\langle \chi \rangle = I \Rightarrow \langle \pi(\chi) \rangle = \pi(I)$.

Classifying $\mathbb{N}$-gradable weight $L_k(\mathfrak{g})$-modules

- A $V_k(\mathfrak{g})$-module M is a $L_k(\mathfrak{g})$-module. $\iff$ I annihilates M.
- Simple $\mathbb{N}$-gradable $L_k(\mathfrak{g})$-modules $\xleftrightarrow{\text{1-1}}$ simple $\frac{U(\mathfrak{g})}{\pi(I)}$-weight modules.
- $U(\mathfrak{g})$-weight modules $\Rightarrow$ $\frac{U(\mathfrak{g})}{\pi(I)}$-weight modules $\Rightarrow$ $\mathbb{N}$-gradable $L_k(\mathfrak{g})$-modules.

Example $\mathfrak{g} = \mathfrak{sl}(2) = \mathrm{span}\{E, F, H\}$

Theorem [Gabriel]

Any simple $\mathfrak{sl}(2)$ weight module with finite dimensional weight spaces is isomorphic to one of the following:

- Finite-dimensional modules $\mathscr{F}_\lambda$, $\lambda \in \mathbb{Z}_{\geq 0}$. Highest and lowest weight. Weights: $\lambda, \lambda - 2, \ldots, 2 - \lambda, -\lambda$
- Infinite-dimensional highest weight modules $\mathscr{H}_\lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Weights: $\lambda, \lambda - 2, \lambda - 4, \ldots$
- Infinite-dimensional lowest weight modules $\mathscr{L}_\lambda$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Weights: $\ldots, \lambda + 4, \lambda + 2, \lambda$
- Infinite-dimensional weight modules $\mathscr{W}_{\lambda;\Delta}$, $\lambda, \Delta \in \mathbb{C}$ and $2\Delta \neq \mu(\mu + 2)$ for any $\mu \in \lambda + 2\mathbb{Z}$, where Δ is the eigenvalue of the quadratic Casimir and $\mathscr{W}_{\lambda;\Delta} \cong \mathscr{W}_{\lambda+2;\Delta}$. Neither highest nor lowest weight.
  Weights: $\ldots, 2 + \lambda, \lambda, \lambda - 2, \ldots$

All weight spaces are 1 dimensional.

Example $\mathfrak{g} = \mathfrak{sl}(2) = \mathrm{span}\{E, F, H\}$

- For $k \in \mathbb{Z}_{\geq 0}$, the ideal of $V_k(\mathfrak{sl}(2))$ is generated by the singular vector $\chi = (E_{-1})^{k+1}\mathbf{1}_k$.
- In $A(V_k(\mathfrak{sl}(2))) \simeq U(\mathfrak{sl}(2))$, we have $\pi((E_{-1})^{k+1}\mathbf{1}_k) = E^{k+1}$.
- The generator E is nilpotent in $A\left(\frac{V_k(\mathfrak{sl}(2))}{\langle E^{k+1}\mathbf{1}_k\rangle}\right) \simeq \frac{U(\mathfrak{sl}(2))}{\langle E^{k+1}\rangle}$.
- The simple $\mathbb{N}$-gradable $\frac{U(\mathfrak{sl}(2))}{\langle E^{k+1}\rangle}$-weight modules are the simple $V_k(\mathfrak{sl}(2))$-weight modules with top grade $\mathscr{F}_\lambda$, $\lambda = 0, \ldots, k$.

Upshot

Easy if the singular vector is easy, very hard if not.

General admissible levels

Let

$$k+2=\frac{u}{v},\quad \lambda_{r,s}=r-1-s\frac{u}{v},\quad \Delta_{r,s}=\frac{r^2-1}{2}+\frac{s^2}{2}\frac{u^2}{v^2}-rs\frac{u}{v}.$$

Theorem [Adamović, Milas] [Ridout,SW]

Any simple $\mathbb{N}$-gradable $L_k(\mathfrak{sl}(2))$-module is isomorphic to one of the following:

- The simple quotients induced from the finite-dimensional modules $\mathscr{F}_{r-1}$, where $1\le r\le u-1$.
- The simple quotients induced from the infinite-dimensional highest weight modules $\mathscr{H}_{\lambda_{r,s}}$, where $1\le r\le u-1$ and $1\le s\le v-1$.
- The simple quotients induced from the infinite-dimensional lowest weight modules $\mathscr{L}_{-\lambda_{r,s}}$, where $1\le r\le u-1$ and $1\le s\le v-1$.
- The simple quotients induced from the infinite-dimensional weight modules $\mathscr{W}_{\lambda,\Delta_{r,s}}$, where $1\le r\le u-1$ and $1\le s\le v-1$, $2\Delta_{r,s}\neq\mu(\mu+2)$ for all $\mu\in\lambda+2\mathbb{Z}$ and $\mathscr{W}_{\lambda,\Delta_{r,s}}\cong\mathscr{W}_{\lambda,\Delta_{u-r,v-s}}$.

Proof idea: Wakimoto free field realisation

$V_k(\mathfrak{sl}(2))$ is a vertex operator subalgebra of $\{$rank 1 Heisenberg$\} \otimes \{\beta\gamma$-ghosts$\}$.

$$a(z)a(w) \sim \frac{1}{(z-w)^2}, \quad \gamma(z)\beta(w) \sim \frac{1}{z-w}, \quad \beta(z)\beta(w) \sim 0 \sim \gamma(z)\gamma(w).$$

$$\begin{aligned}
E(z) &= \beta(z), \\
H(z) &= 2 : \beta(z)\gamma(z) : + \sqrt{2k+4}\, a(z), \\
F(z) &= : \beta(z)\gamma(z)\gamma(z) : + \sqrt{2k+4} : a(z)\gamma(z) : + k\partial\gamma(z).
\end{aligned}$$

Screening operator

$$S(z) = : \beta(z) \exp\left(-\sqrt{\frac{2}{k+2}}\phi(z) \right) :, \quad \partial\phi(z) = a(z).$$

Proof idea: Wakimoto free field realisation

The singular vector of $V_{\frac{u}{v}-2}(\mathfrak{sl}(2))$, in the free field realisation, can be realised by the screening operator.

$$\begin{aligned} S^{[u-1]} \left|q\right\rangle &= \int S(z_1)\cdots S(z_{u-1}) \left|q\right\rangle \mathrm{d}z \\ &= \int \beta(z_1)\cdots\beta(z_{u-1}) \end{aligned}$$

$$\times \prod_{1\le i\ne j\le u-1} \left(1-\frac{z_i}{z_j}\right)^{\frac{v}{u}} \prod_{i=1}^{u-1} z_i^{-v-1} \prod_{m\ge 1} \exp\left(-\sqrt{\frac{2v}{u}}\frac{\mathsf{p}_m(z)a_{-m}}{m}\right)\left|q\right\rangle \frac{\mathrm{d}z_1\cdots\mathrm{d}z_{u-1}}{z_1\cdots z_{u-1}},$$

where

$$q=(u-1)\sqrt{\frac{2v}{u}}, \quad \mathsf{p}_m(z)=\sum_{i=1}^{u-1} z_i^m.$$

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$$\times \underbrace{\prod_{1\le i\ne j\le u-1}\left(1-\frac{z_i}{z_j}\right)^{\frac{v}{u}}}_{\substack{\text{Inner prod. of} \\ \text{Jack symm. poly.}}} \prod_{i=1}^{u-1} z_i^{-v-1} \prod_{m\ge 1}\exp\left(-\sqrt{\frac{2v}{u}}\frac{\mathsf{p}_m(z)a_{-m}}{m}\right)\left|q\right\rangle \frac{\mathrm{d}z_1\cdots\mathrm{d}z_{u-1}}{z_1\cdots z_{u-1}},$$

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Proof idea: The generator of the ideal

Choose a generator of $\mathfrak{sl}(2)$-weight 0.

$$\chi = F_0^{u-1} S^{[u-1]} |q\rangle = \text{const} \cdot S^{[u-1]} \gamma_0^{u-1} |q\rangle$$

Compute eigenvalue of zero-mode χ_0 on a general top grade vector to determine image in $A(V_{\frac{u}{v}-2}(\mathfrak{sl}(2)))$.

$$\chi_0|p,\tau\rangle = f(p,\tau)|p,\tau\rangle, \quad p = \text{Heisenberg weight}, \ \tau = \beta\gamma\text{-weight}.$$

Theorem [Ridout, SW]

1. The polynomial $f(p,\tau)$, in free field data, is also a polynomial in $\mathfrak{sl}(2)$-data.
2. $$f(\lambda,\Delta) = g_u(\lambda,\Delta) \prod_{r,s} (\Delta - \Delta_{r,s}),$$
$$g_{u+2}(\lambda,\Delta) = \frac{(2u+1)\lambda}{(u+1)^2} g_{u+1}(\lambda,\Delta) - \frac{2\Delta - (u-1)(u+1)}{(u+1)^2} g_u(\lambda,\Delta)$$
$$g_1(\lambda,\Delta) = 1, \quad g_2(\lambda,\Delta) = \lambda.$$

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$$A(L_k(\mathfrak{sl}(2))) = \frac{U(\mathfrak{sl}(2))}{\langle f(H,Q) \rangle}$$

Outlook

Computing the Zhu algebra becomes almost algorithmic when using free fields and symmetric polynomials. Successfully applied to:

- Virasoro minimal models
- $\mathfrak{sl}(2)$ minimal models
- $W_{p,q}$-triplet models

Work in progress:

- $N = 1$ Virasoro minimal models
- $\mathfrak{osp}(1|2)$ minimal models

Future plans:

- higher rank affine Kac-Moody superalgebras
- higher rank W-algebras.

The End

Thank you!