Maximal green sequences of minimal mutation-infinite quivers

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**Theorem.** All minimal mutation-infinite quivers have a maximal green sequence.

**Theorem.** Any cluster algebra generated by a minimal mutation-infinite quiver is equal to its upper algebra.

**Theorem.** The different move-classes of minimal mutation-infinite quivers belong to different mutation-classes (mostly...).
(Cluster) quiver — directed graph with no loops or 2-cycles.

**Mutation** $\mu_k$ at vertex $k$:

- Add arrow $i \to j$ for each path $i \to k \to j$
- Reverse all arrows adjacent to $k$
- Remove maximal collection of 2-cycles

**Induced subquiver** — obtained by removing vertices.
Quivers and mutations

Quiver $Q$ is **mutation-equivalent** to $P$ if there are mutations taking $Q$ to $P$.

Mut($Q$) is the **mutation class** of $Q$ containing all quiver mutation-equivalent to $Q$.

$Q$ is **mutation-finite** if its mutation class is finite. Otherwise it is **mutation-infinite**.

$Q$ is **minimal mutation-infinite** if every induced subquiver is mutation-finite.
MMI classes

Minimal mutation-infinite quivers classified into move-classes [L ’16], with representatives:

• Hyperbolic Coxeter simplex representatives
• Double arrow representatives
• Exceptional representatives
Hyperbolic Coxeter simplex diagrams
Double arrow representatives
Exceptional type representatives
A framed quiver $\hat{Q}$ is constructed from quiver $Q$, by adding an additional frozen vertex $\hat{i}$ for each vertex $i$ in $Q$ and a single arrow $i \to \hat{i}$. 

Framed quivers
A mutable vertex $i$ in $\hat{Q}$ is **green** if there are no arrows $\hat{j} \rightarrow i$.

A mutable vertex $i$ in $\hat{Q}$ is **red** if there are no arrows $i \rightarrow \hat{j}$.

**Theorem (Derksen-Weyman-Zelevinsky ’10).** Any mutable vertex in a quiver is red or green.
Maximal green sequences

Assume a quiver $Q$ has vertices labelled $(1, \ldots, n)$.

A **mutation sequence** is a sequence of vertices $i = (i_1, \ldots, i_k)$ corresponding to mutating first in vertex $i_1$, then $i_2$ and so on.

A **green sequence** is a mutation sequence where every mutation is at a green vertex.

A **maximal green sequence** is a green sequence where every mutable vertex in the resulting quiver is red.
MGS example
**Some results**

**Proposition (Brüstle-Dupont-Perotin ’14).** If $i$ is a maximal green sequence for $Q$ then $\mu_i(Q)$ is isomorphic to $Q$.

The **induced permutation** of a maximal green sequence is the permutation $\sigma$ such that $\sigma(\mu_i(Q)) = Q$.

**Theorem (BPS ’14).** Any acyclic quiver has a maximal green sequence.

**Proposition (BPS ’14).** A quiver $Q$ has a maximal green sequence if and only if $Q^{\text{op}}$ has a maximal green sequence.
More results

**Proposition (Muller ’15).** If $Q$ has a maximal green sequence, every induced subquiver has a maximal green sequence.

**Proposition (Muller ’15).** Having a maximal green sequence is not mutation-invariant.

**Proposition (Mills ’16).** If $Q$ is a mutation-finite quiver, then provided $Q$ does not arise from a once-punctured closed surface and is not mutation-equivalent to the type $X_7$ quiver, then $Q$ has a maximal green sequence.
Lemma (Brüstle-Hermes-Igusa-Todorov ’15). If $i = (i_1, i_2, \ldots, i_\ell)$ is a maximal green sequence of $Q$ with induced permutation $\sigma$, then $(i_2, \ldots, i_\ell, \sigma^{-1}(i_1))$ is a maximal green sequence for the quiver $\mu_{i_1}(Q)$ with the same induced permutation.

Lemma. If $i = (i_1, \ldots, i_{\ell-1}, i_\ell)$ is a maximal green sequence of $Q$ with induced permutation $\sigma$, then $(\sigma(i_\ell), i_1, \ldots, i_{\ell-1})$ is a maximal green sequence for the quiver $\mu_{\sigma(i_\ell)}(Q)$ with the same induced permutation.
Direct sums of quivers
[Garver-Musiker ’14]

Given two quivers $P$ and $Q$ with $k$-tuples $(a_1, \ldots, a_k)$ of vertices of $P$, $(b_1, \ldots, b_k)$ of vertices of $Q$, the **direct sum**

$$P \oplus_{(a_1, \ldots, a_k)} (b_1, \ldots, b_k) Q$$

is the quiver obtained from the disjoint union of $P$ and $Q$, with additional arrows $a_i \to b_i$ for each $i$.

This is a **$t$-coloured direct sum** if $t$ is the number of distinct vertices in $(a_i)$ and there are no repeated arrows $a_i \to b_j$ added.
MGS for direct sums

Theorem (GM '14). If $P = Q \oplus_{(a_1, \ldots, a_k)}^{(b_1, \ldots, b_k)} R$ is a $t$-colored direct sum, $(i_1, \ldots, i_r)$ is a maximal green sequence for $Q$, and $(j_1, \ldots, j_s)$ is a maximal green sequence for $R$, then

$$(i_1, \ldots, i_r, j_1, \ldots, j_s)$$

is a maximal green sequence for $P$. 
Quivers ending in a 3-cycle

Theorem. If $Q$ ends in a 3-cycle and $C$ has a maximal green sequence $i_C$, then $Q$ has a maximal green sequence $(b, i_C, a, b)$. 
Rank 3 MMI quivers
and maximal green sequences

Proposition (Muller '15).
If \(a, b\) and \(c \geq 2\) then \(Q_{a,b,c}\) does not have a maximal green sequence.

Proposition. If any of \(a, b\) or \(c\) are 1, then \(Q_{a,b,c}\) has a maximal green sequence.
Higher ranks

Recall: all mutation-finite quivers have a maximal green sequence, unless they come from a triangulation of a once-punctured closed surface or are mutation-equivalent to $X_7$.

**Lemma.** No minimal mutation-infinite quiver contains a subquiver which does not have a maximal green sequence.

**Corollary.** Every subquiver of a minimal mutation-infinite quiver has a maximal green sequence.
**Theorem.** If $Q$ is a minimal mutation-infinite quiver of rank at least 4 then $Q$ has a maximal green sequence.

Most have a sink or a source — leaving 192.

Many others are direct sums — leaving 42.

35 of these end in a 3-cycle — leaving 7.
The remaining 7 quivers
Moves are sequences of mutations.

Quivers in the same class must be mutation-equivalent.

But does each move-class belong to a different mutation-class?
Rank of the adjacency matrix is mutation-invariant [Berenstein-Fomin-Zelevinsky ’05].

Determinant of the adjacency matrix is mutation-invariant.

Whether a quiver is mutation-acyclic — and how many acyclic quivers are in the mutation class [Caldero-Keller ’06].
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How can you prove that a quiver is not mutation-equivalent to an acyclic quiver?

Use the idea of admissible quasi-Cartan companions.
A \textbf{quasi-Cartan companion} of a quiver \( Q \) is a symmetric matrix \( A = (a_{i,j}) \) such that \( a_{i,i} = 2 \) and \( a_{i,j} = |b_{i,j}| \) where \( B = (b_{i,j}) \) is the adjacency matrix of \( Q \).

A quasi-Cartan companion of \( Q \) is \textbf{admissible} if for any oriented (resp., non-oriented) cycle \( Z \) in \( Q \), there are an odd (resp., even) number of edges \( \{i,j\} \) in \( Z \) such that \( a_{i,j} > 0 \).

\textbf{Theorem (Seven ‘15).} If \( Q \) is mutation-acyclic, then \( Q \) has an admissible quasi-Cartan companion.
How can you prove a quiver does not have an admissible quasi-Cartan companion?

*Proposition (Seven ’11).* Two admissible companions of a quiver $Q$ can be obtained from one another by a number of simultaneous sign changes in rows and columns.
**Corollary.** This quiver is not mutation-acyclic.
**Proposition.** Each double arrow move-class contains no acyclic quivers.

Each representative is mutation-equivalent to something which contains:

![Diagram](attachment:image.png)
Example

(3, 4, 5, 6)
**Proposition.** Each exceptional move-class contains no acyclic quivers.

But don’t know if they belong to different mutation-classes to each other or to the double arrow classes.