NP-Complete Problems

You Complete Me...
Overview

What we have proved so far...

Cook-Levin Theorem:
SAT is NP-Complete

SAT ≤_p 3SAT
therefore 3SAT is NP-Complete.

3SAT ≤_p CLIQUE
therefore CLIQUE is NP-Complete
Theorem: Vertex Cover

If $G$ is an undirected graph, a vertex cover of $G$ is a subset of the nodes where every edge of $G$ touches one of those nodes. The vertex cover problem asks whether a graph contains a vertex cover of a specified size:

$$\text{VERTEX-COVER} = \{ \langle G, k \rangle \mid G \text{ is an undirected graph that has a k-node vertex cover} \}.$$ 

THEOREM: “Vertex Cover is NP-Complete”

To show it is NP, define the certificate as the vertex cover of size $k$, which can be verified in polynomial time. Easy.

To show that all NP-Complete problems are polynomial-time reducible to it.

IDEA: Reduce 3SAT $\varphi$ to VERTEX-COVER.

In effect, $G$ will simulate $\varphi$.

Must create gadgets that mimic the variables and clauses of $\varphi$.

A gadget is a construction that mimics some functionality for the purpose of the reduction.
Vertex Cover Gadgets Construction

**Variable Gadget**
- Represent the true and false state of each variable, separated with an edge.

**Clause Gadget**
- Group of 3 nodes.
- Vertex cover must contain 2 of the 3 nodes.

Choose $k$ so that vertex cover has one node per variable gadget and two nodes per clause gadget.

Choose $k = 8$ such that:

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (x_1 \lor x_2 \lor x_2)$$
**Vertex Cover Proof**

For **vertex** gadget:
- In $\varphi$, setting $x$ to **TRUE** corresponds to selecting node labeled $x$ in the gadget.
- In $\varphi$, setting $x$ to **FALSE** corresponds to selecting node labeled $\neg x$ in the gadget.
- For either value, edge between $x$ and $\neg x$ will always touch a node in the vertex cover.
- 1 node from each gadget must be included in the vertex cover.

For **clause** gadget:
- Each node of the gadget has 3 edges.
  ○ 2 edges connect to other nodes in the gadget
  ○ 1 edge connects to a vertex gadget
- Selecting any 2 nodes of the gadget will result in all 3 internal edges being touched.
  ○ 2 nodes from each clause gadget must be included in the vertex cover.

- Each of the 3 nodes of the gadget has an edge to a vertex gadget.
- An edge connects a vertex gadget node to a clause gadget node.
  ○ If the clause is false, then no edge will touch a vertex cover node in any vertex gadget. Because only 2 vertices are in the vertex cover in the vertex gadget, 1 edge will not be touching the vertex cover.
  ○ If the clause is true, then at least one edge will touch a vertex cover node in the vertex gadget, and the other two edges may touch the other two vertex cover nodes in the clause gadget.
Other NP-Complete Problems

In The Book:
- Hamiltonian Path
- Undirected Hamiltonian Path

Be careful: Traveling Salesman has 2 forms:
1. A decision problem (NP-Complete)
2. An optimization problem (NP-Hard)

Super Mario Brothers!!
What else is there?

Larger classes

Probabilistic categories

More involved theoretical analysis