Definitions & Theorems & Proofs

There exists some number $x$ such that $f(x) = g(f(0)) = 1$.

Oh yes, somewhere out there, it exists.

And we must find it... and destroy it.

Grab your swords, students! We ride!

I think I'm in the wrong math class?

I'm finally in the right one.
Definitions

Yes, we are actually defining it.

Examples: $B = \{w \# w \mid w \in \{0, 1\}^*\}$

$L = \{0^n1^n \mid n \geq 0\}$

A **Definition** describes an object, notation, or idea.

Often used in subsequent mathematical statements.

Standard (mathematical) symbols do not need to be defined:

- **Traditional notation**
  - Set Operations ($\emptyset, \cup$)
  - Arithmetic/Logical Operators ($\land, \neg$)

- **Sets of numbers** ($\mathbb{R}, \mathbb{N}$)

- **Standard functions** ($\log, \sqrt{}$)

When in doubt, define it.
Proof - A convincing logical argument that a statement is true.

Theorem - A mathematical statement proven true.

Lemma - Smaller theorem, used to prove other, larger theorems.

Corollary - Statement that is related to a theorem that is trivial to show, given the theorem is true.
Example

Theorem: For every graph $G$, the sum of the degrees of all the nodes in $G$ is an even number.

Definitions:
- $G = (V, E)$, where $V$ is a set of vertices (nodes) and,
- $E = \{(u, v) \mid u, v \in V\}$ is the set of edges.

1. What proof would you give?
2. What does the definition of $E$ imply about the graph?
3. Does it matter?
Of course, proofs can be tricky...
3 Types Of Proofs

That we will use in this class.

**Construction**
- Proves that something exists by showing a (generalized) method to construct it, then demonstrates that the method is correct.

**Contradiction**
- Assumes the theorem is false, then uses logical argument to show that the assumption leads to a false consequence, thereby proving the original theorem to be true.

**Induction**
- Shows that all members of an infinite set have a specified property using a basis and an induction step.
Example: Proof by Construction

**Definition:** We define a graph to be \( k \)-regular if every node in the graph has degree \( k \).

**Theorem:** For each even number \( n \) greater than 2, there exists a 3-regular graph with \( n \) nodes.

**Proof:** Let \( n \) be an even number greater than 2. Construct graph \( G = (V, E) \) with \( n \) nodes as follows. The set of nodes of \( G \) is \( V = \{0, 1, ..., n-1\} \), and the set of edges of \( G \) is the set

\[
E = \{(i, i+1) \mid 0 \leq i \leq n-2\} \cup \{(n-1, 0)\} \cup \{(i, i + n/2) \mid 0 \leq i \leq (n/2) - 1\}.
\]

In the description of \( E \), the first two sets form a closed circle, with each node having degree 2. The 3rd set is an edge across the diameter of the circle, providing the 3rd degree for each node in \( V \).

Therefore, every node in \( G \) will have exactly 3 degrees. \( \square \)

**Parts of Proof:**
- Construction
- Proof of correctness
- Recap
**Example: Proof by Contradiction**

**Theorem:** $\sqrt{2}$ is Irrational.

**Proof:** Assume that $\sqrt{2}$ is rational. Therefore it must be expressible as a fraction $m/n$, where $m$ and $n$ are integers and $m/n$ is a fraction reduced to lowest terms.

If reduced to lowest terms, either $m$ or $n$ must be odd.

Algebraic manipulation yields:

\[
\begin{align*}
\sqrt{2} &= m/n \\
n\sqrt{2} &= m \\
(n\sqrt{2})^2 &= m^2 \\
n^2 \cdot 2 &= m^2
\end{align*}
\]

Because $m^2$ is 2 times the integer $n^2$, we know that $m^2$ is even. Therefore, $m$, too, is even, as the square of an odd number always is odd. So we can write $m = 2k$ for some integer $k$. Then, substituting $2k$ for $m$, we get:

\[
\begin{align*}
2n^2 &= (2k)^2 \\
2n^2 &= 4k^2 \\
n^2 &= 2k^2
\end{align*}
\]

This, however, shows that $n^2$ is even, and thus $n$ is even. Earlier, we had reduced both $m$ and $n$ so that they are not both even. This is a contradiction. The assumption must be false. $\blacksquare$
Example: Proof by Induction

Theorem: $P(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Base Case: $P(0) = \frac{0 \cdot (0 + 1)}{2} = 0$

Induction Step:

Assume that $P(k)$ is true. Show that if $P(k)$ is true, then $P(k+1)$ holds.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$$

Using the inductive hypothesis that $P(k)$ is true, the left hand side may be rewritten as:

$$\frac{k(k+1)}{2} + (k + 1)$$

Algebraic manipulation on the left hand side yields:

$$\frac{k(k+1)}{2} + (k + 1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

The left hand side now matches the right hand side of the equation, thereby showing that $P(k+1)$ does hold.

Since the base case and inductive step are shown to be true, the theorem is shown to hold. 

What is missing?
Find The Error

CLAIM: In any set of \( h \) horses, all horses are the same color.

PROOF: By induction on \( h \).

Basis: For \( h = 1 \). In any set containing just one horse, all horses clearly are the same color.

Induction step: For \( k \geq 1 \), assume that the claim is true for \( h = k \) and prove that it is true for \( h = k+1 \). Take any set \( H \) of \( k+1 \) horses. We show that all the horses in this set are the same color.

Remove one horse from this set to obtain the set \( H_1 \) with just \( k \) horses. By the induction hypothesis, all the horses in \( H_1 \) are the same color. Now replace the removed horse and remove a different one to obtain the set \( H_2 \). By the same argument, all the horses in \( H_2 \) are the same color.

Therefore, all the horses in \( H \) must be the same color, and the proof is complete. \( \square \)
This is a Sea Horse.

Your argument is invalid.
Homework Expectations

1. **Definitions**: All new symbols must be defined within the scope of your proof.

2. **Must provide the following parts**:
   a. **Problem Statement**: Describe what you are trying to solve.
   b. **Intuition Statement**: Describe your approach informally, so that the grader can more easily understand your logic. Can be mostly prose.
   c. **Proof**: Include all necessary parts. Proof must read as a stand-alone statement, and should use mathematical formalisms where necessary for clarity & conciseness. Include closing statement.
      i. **Construction**: Must show construction & correctness of construction
      ii. **Contradiction**: Must state assumption & show contradiction
      iii. **Induction**: Must show Basis step & Induction step

3. **Beware!**
   a. Edge cases
   b. Iff must show both directions!
My professor got fed up trying to explain what a theorem is and decided to try a different approach.