Math 16B, Spring '11
Quiz 10, April 12

1. (5 points) Use the integral test to determine whether the series

$$\sum_{k=1}^{\infty} ke^{-k^2}$$

is convergent or divergent.

Solution. We can write our series as

$$\sum_{k=1}^{\infty} f(k),$$

where

$$f(x) = x \cdot e^{-x^2}.$$

f is a continuous function, being the product of the continuous functions x and e^{-x^2}. It is positive for $x \geq 1$, because in this case $x > 0$, while the exponential term is always positive. To see that f is decreasing, we take the derivative of f (using the product rule and the chain rule):

$$f'(x) = e^{-x^2} + x \cdot (e^{-x^2} \cdot (-2x)) = e^{-x^2} - 2x^2 e^{-x^2} = (1 - 2x^2)e^{-x^2} < 0$$

because $1 - 2x^2 \leq 1 - 2 = -1 < 0$ (for $x \geq 1$), and $e^{-x^2} > 0$. We can thus apply the Integral Test. Our series is convergent if and only if the improper integral

$$\int_1^{\infty} xe^{-x^2} dx = \lim_{b \to \infty} \int_1^b xe^{-x^2} dx \qquad (*)$$

converges. We first compute the indefinite integral

$$\int xe^{-x^2} dx$$

using the substitution $u = -x^2$. This yields $du = -2xdx$, i.e. $xdx = -du/2$ and

$$\int xe^{-x^2} dx = \frac{-1}{2} \int e^u du = \frac{-1}{2} \cdot e^{-x^2} + C.$$

It follows that

$$\int_1^b xe^{-x^2} dx = \left[\frac{-1}{2} \cdot e^{-x^2}\right]_1^b = \frac{e^{-1} - e^{-b^2}}{2}.$$

As $b \to \infty$, $-b^2 \to -\infty$, hence $e^{-b^2} \to 0$. It follows that

$$\lim_{b \to \infty} \int_1^b xe^{-x^2} dx = \lim_{b \to \infty} \frac{e^{-1} - e^{-b^2}}{2} = \frac{1}{2e},$$

which is finite. The improper integral is thus convergent, hence also is our series. $\square$

2. Find the Taylor series of $\ln(1 + x^2)$ at $x = 0$.

Solution. Since the Taylor series of $\ln(1-x)$ at $x=0$ is

$$\sum_{k=1}^{\infty} -\frac{x^k}{k} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots,$$

we can obtain the Taylor series of $\ln(1+x^2)$ at $x=0$ by substituting x by $-x^2$ in the above formula. We get that the Taylor series of $\ln(1+x^2)$ is equal to

$$\sum_{k=1}^{\infty} -\frac{(-x^2)^k}{k} = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^{2k}}{k} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$$

□