Math 16B, Spring '11
Quiz 2, February 1

1. (5 points) Find all points (x, y) where

$$f(x, y) = x^2 - 4xy + 2y^4$$

has a possible relative maximum or minimum. Then use the second-derivative test to determine, if possible, the nature of $f(x, y)$ at each of these points. If the second-derivative test is inconclusive, so state.

Solution. We first find the points for which the partial derivatives of f vanish:

$$\frac{\partial f}{\partial x} = 2x - 4y = 0,$$

$$\frac{\partial f}{\partial y} = -4x + 8y^3 = 0.$$

The first equation yields $x = 2y$. Substituting x in the second equation, we obtain

$$-8y + 8y^3 = 0 \Leftrightarrow y^3 = y,$$

i.e. $y = 0, 1$ or -1, and the corresponding values for x are, respectively, $0, 2$ and -2.

To apply the second-derivative test, we need the partial derivatives of 2nd order for f:

$$\frac{\partial^2 f}{\partial x^2} = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4,$$

$$\frac{\partial^2 f}{\partial y^2} = 24y^2.$$

We get

$$D(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 48y^2 - 16.$$

For $(x, y) = (0, 0)$, $D(0, 0) < 0$ so we get a saddle point.

For $(x, y) = (2, 1)$ and $(x, y) = (-2, -1)$, $D(x, y) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, hence we get relative minima at these points.

$\square$

2. (5 points) The amount of space required by a particular firm is

$$f(x, y) = 1000\sqrt{6x^2 + y^2},$$

where x and y are, respectively, the number of units of labor and capital utilized. Suppose that labor costs \$480 per unit and capital costs \$40 per unit and that the firm has \$5000 to spend. Determine the amounts of labor and capital that should be utilized in order to minimize the amount of space required.

Solution. The points (x, y) that minimize the value of f are the same as those that minimize the value of $\sqrt{6x^2 + y^2}$ (since this is just a scalar multiple of f), and furthermore, the same as those that minimize $6x^2 + y^2$ (since a positive function is minimized at the same points as its square).

We seek to minimize $6x^2 + y^2$ under the assumption that $480x + 40y = 5000$ ($C(x, y) = 480x + 40y$ represents the cost of utilizing x units of labor and y units of capital). We use the method of Lagrange Multipliers. Let

$$F(x, y, \lambda) = 6x^2 + y^2 + \lambda(480x + 40y - 5000).$$

We impose the condition that the partial derivatives of F are 0:

$$\frac{\partial F}{\partial x} = 12x + 480\lambda = 0,$$

$$\frac{\partial F}{\partial y} = 2y + 40\lambda = 0,$$

$$\frac{\partial F}{\partial \lambda} = 480x + 40y - 5000 = 0.$$

The first equation yields $x = -40\lambda$, while the second one yields $y = -20\lambda$. It follows that $x = 2y$, which combined with the last equation yields

$$960y + 40y - 5000 = 0, \text{ i.e. } y = 5, \text{ and } x = 10.$$

The minimum value of $f(x, y)$ is thus attained at $(10, 5)$ and is equal to

$$f(10, 5) = 1000\sqrt{6 \cdot 10^2 + 5} = 1000\sqrt{625} = 25000.$$

$\square$