Math 16B, Spring '11
Quiz 7, March 15

1. (5 points) A person took out a loan of \$100,000 from a bank that charghes 7.5% interest compounded continuously. What should be the annual rate of payments if the loan is to be paid in full in exaclty 10 years? (Assume that the payments are made continuously throughout the year)

Solution. Let r denote the anual rate of payment for which the loan is to be paid in full in exactly 10 years, and let $y(t)$ denote the amount of debt after t years. We have

$$y(0) = 0, \quad y(10) = 100000,$$

and y satisfies the differential equation

$$y' = 0.075y - r.$$

We put the equation into the form

$$y' + a(t)y = b(t),$$

where $a(t) = -0.075$, $b(t) = -r$. We find $A(t) = -0.075t$ and

$$y \cdot e^{-0.075t} = \int e^{-0.075t} \cdot (-r)dt = \frac{e^{-0.075t}}{-0.075} \cdot (-r) + C,$$

i.e.

$$y = \frac{r}{0.075} + Ce^{0.075t}.$$

The conditions $y(0) = 100000$ and $y(10) = 0$ yield

$$100000 = \frac{r}{0.075} + C,$$

$$0 = \frac{r}{0.075} + C \cdot e^{0.75}.$$

We eliminate C by multiplying the first equality by $e^{0.75}$ and subtracting the second. This yields

$$100000e^{0.75} = \frac{r}{0.075}(e^{0.75} - 1),$$

i.e.

$$r = \frac{7500e^{0.75}}{e^{0.75} - 1} \simeq 14214.$$

□

2. (5 points) For information being spread by mass media, rather than through individual contact, the rate of spread of the information at any time is proportional to the percentage of the population not having the information at that time. Give the differential equation that is satisfied by $y = f(t)$, the percentage of the population having the information at time t. Assume that $f(0) = 1\%$. Sketch the solution.

Proof. The rate of spread of the information is measured by y', while the percentage of the population not having the information is computed by $(1 - y)$. The two quantities being proportional means that

$$y' = k(1 - y) \tag{*}$$

for some constant k. Note that since the rate of spread of information is positive, and $(1 - y)$ is positive, k has to also be positive.

To sketch the graph of the solution curve corresponding to $y(0) = 1\% = 0.01$ we first notice that it's bounded above by the graph of the unique constant solution of $(*)$, the line $y = 1$. Now $y'(0.01) = k \cdot 0.99 > 0$ so the solution curve is increasing. $y'' = -ky'$ is negative, because both k and y' are positive, hence the solution curve is concave. To summarize: the solution curve is increasing, concave, with asymptote the line $y = 1$ (sketch it!). □