Math 16B, Spring '11
Quiz 8, March 29

1. (5 points) Evaluate
$$\int_1^e \ln(x)dx.$$

Solution. We first compute the indefinite integral $\int \ln(x)dx$. We write $\ln(x)$ as the product $1 \cdot \ln(x)$ and use integration by parts. Since 1 is an algebraic function and $\ln(x)$ is logarithmic, LIATE tells us to choose
$$f(x) = \ln(x), \quad g(x) = 1.$$
We get
$$f'(x) = \frac{1}{x}, \quad G(x) = x$$
and therefore
$$\int \ln(x)dx = x\ln(x) - \int \frac{1}{x} \cdot xdx = x\ln(x) - x + C.$$
It follows that the value of the definite integral is given by
$$\int_1^e \ln(x)dx = [x\ln(x) - x]_1^e = e\ln(e) - e - (1\ln(1) - 1) = 1.$$
□

2. (5 points) Solve the differential equation
$$\frac{1}{\sqrt{t+1}}y' + y = 1.$$

First solution. We rewrite the equation as
$$\frac{1}{\sqrt{t+1}} \cdot \frac{dy}{dt} = 1 - y,$$
and separate the variables. We obtain
$$\frac{dy}{1-y} = \sqrt{t+1}dt, \tag{*}$$
unless $1 - y = 0$ (in which case we shouldn't be dividing by $0 = 1 - y$).

If $1 - y = 0$, or equivalently $y = 1$, we easily see that we obtain a (constant) solution for our equation.

Assuming $y \neq 1$, and integrating the equality (*) we get
$$\int \frac{dy}{1-y} = \int \sqrt{t+1}dt,$$
or equivalently
$$-\ln|1-y| = \frac{2}{3} \cdot (t+1)^{3/2} + C.$$

Multiplying both sides by -1 and exponentiating, this yields

$$|1-y| = e^{-\frac{2}{3}\cdot(t+1)^{3/2}-C} = e^{\frac{2}{3}\cdot(t+1)^{3/2}} \cdot e^{-C},$$

i.e.

$$1-y = \pm e^{-C} \cdot e^{-\frac{2}{3}\cdot(t+1)^{3/2}}.$$

We denote the nonzero constant $\pm e^{-C}$ by A, and obtain

$$y = 1 - A \cdot e^{-\frac{2}{3}\cdot(t+1)^{3/2}}.$$

Observing that $A=0$ yields the solution $y=1$ obtained earlier, we conclude that it's okay to allow A to take the value zero also, and hence all the solutions of our original equation are given by the formula

$$y = 1 - A \cdot e^{-\frac{2}{3}\cdot(t+1)^{3/2}},$$

where A is an arbitrary constant. □

Second solution. We treat our equation as a first order linear differential equation. Multiplying both sides by $\sqrt{t+1}$ yields

$$y' + \sqrt{t+1} \cdot y = \sqrt{t+1}, \qquad (**)$$

i.e. in the usual notation

$$a(t) = b(t) = \sqrt{t+1}.$$

We have

$$A(t) = \int a(t)dt = \frac{2}{3}(t+1)^{3/2},$$

hence the integrating factor is given by

$$I(t) = e^{A(t)} = e^{\frac{2}{3}(t+1)^{3/2}}.$$

Multiplying both sides of (**) by $I(t)$ and integrating, we obtain

$$y \cdot e^{A(t)} = \int e^{A(t)} \cdot b(t)dt,$$

i.e.

$$y \cdot e^{\frac{2}{3}(t+1)^{3/2}} = \int e^{\frac{2}{3}(t+1)^{3/2}} \cdot \sqrt{t+1}dt.$$

We make the substitution $u = A(t)$ yields $du = \sqrt{t+1}dt$

$$y \cdot e^{\frac{2}{3}(t+1)^{3/2}} = \int e^u du = e^u + C = e^{\frac{2}{3}(t+1)^{3/2}} + C.$$

Dividing by $e^{\frac{2}{3}(t+1)^{3/2}}$ yields

$$y = 1 + Ce^{-\frac{2}{3}(t+1)^{3/2}}$$

where C is an arbitrary constant. □