Review Session - Midterm 1

Claudiu Raicu

September 26, 2009

7 Techniques of integration

The basic formula that you need to remember is

$$\int u dv = uv - \int v du,$$

or if $u = f(x), v = g(x)$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Example 7.1. *Evaluate*

$$\int x^2 \cos(2x)dx.$$

Solution. The general rule to integrate a product $P(x)T(x)$ between a polynomial function and a trigonometric function is to use integration by parts with $u = P(x)$, $dv = T(x)dx$. This will decrease the degree of the polynomial, and you should repeat the procedure as long as the polynomial has positive degree.

In our example, $P(x) = x^2$ and $T(x) = \cos(2x)$. Integration by parts with $u = x^2$ and $dv = \cos(2x)dx$ yields $du = 2xdx$, $v = \frac{1}{2}\sin(2x)$, so

$$\int x^2 \cos(2x)dx = \int u dv = uv - \int v du = x^2 \frac{\sin(2x)}{2} - \int x \sin(2x)dx. \tag{1}$$

Now $x\sin(2x)$ is again a product between a polynomial and a trigonometric function, but the new polynomial has smaller degree, so we've made progress. To evaluate $\int x\sin(2x)dx$, we use again integration by parts, with $u = x$, $dv = \sin(2x)$, which yields $du = dx$, $v = \frac{-\cos(2x)}{2}$. We get

$$\begin{aligned}\int x\sin(2x)dx &= \int u dv = uv - \int v du \\ &= -x\frac{\cos(2x)}{2} + \frac{1}{2}\int \cos(2x)dx \\ &= -x\frac{\cos(2x)}{2} + \frac{\sin(2x)}{4} + C.\end{aligned} \tag{2}$$

Combining (1) and (2) we get

$$\int x^2\cos(2x)dx = x^2\frac{\sin(2x)}{2} + x\frac{\cos(2x)}{2} - \frac{\sin(2x)}{4} + C = \frac{2x^2-1}{4}\sin(2x) + \frac{x}{2}\cos(2x) + C.$$

□

Exercise 7.2. *Try*

$$\int x^3\cos(x)dx.$$

Answer:

$$(x^3 - 6x)\sin(x) + (3x^2 - 6)\cos(x).$$

Many times you may be required to make a substitution to reduce to the above case.

Example 7.3. *Evaluate*

$$\int x\cos(x^{2/3})dx.$$

Proof. The substitution $u = x^{2/3}$ yields $u^3 = x^2$, or after differentiation $3u^2du = 2xdx$. We can therefore replace the term xdx by $\frac{3}{2}u^2du$, and $\cos(x^{2/3})dx$ to get

$$\int x\cos(x^{2/3})dx = \frac{3}{2}\int u^2\cos(u)du.$$

Now apply the above strategy to finish the problem. The final answer should be

$$3x^{2/3}\cos(x^{2/3}) + \frac{3(x^{4/3}-2)}{2}\sin(x^{2/3}) + C.$$

□

We can summarize the above examples in the following pair of *reduction formulas*:

Example 7.4. *Show that*

$$\int x^n\cos(x)dx = x^n\sin(x) - n\int x^{n-1}\sin(x)dx. \tag{3}$$

$$\int x^n\sin(x)dx = -x^n\cos(x) + n\int x^{n-1}\cos(x)dx. \tag{4}$$

Solution. To prove (3), follow the above strategy: let $u = x^n, dv = \cos(x)dx$, $du = nx^{n-1}dx$, $v = \sin(x)$. We get

$$\int x^n\cos(x)dx = \int udv = uv - \int vdu = x^n\sin(x) - n\int x^{n-1}\sin(x)dx.$$

□

Exercise 7.5. *Prove (4).*

The above discussion can be easily translated into a strategy for integrating $P(x)e^x$ where P is a polynomial function.

Example 7.6. *Prove the reduction formula*

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

Proof. We use integration by parts with $u = x^n$, $dv = e^x dx$, which yields $du = nx^{n-1}dx$, $v = e^x$, and

$$\int x^n e^x dx = \int u dv = uv - \int v du = x^n e^x - n \int x^{n-1} e^x dx.$$

$\square$

Exercise 7.7. *Evaluate*

$$\int x^3 e^x dx.$$

(See Exercise 7.2.)

Exercise 7.8. *Evaluate*

$$\int x^2 e^{2x} dx.$$

(See Example 7.1.)

Exercise 7.9. *Evaluate*

$$\int x e^{x^{2/3}} dx.$$

(See Example 7.3.)

Substitutions. It is a general rule that whenever the integrand involves composition of functions like $\sin(f(x))$, $\tan(f(x))$, $e^{f(x)}$, $\sqrt{1 - f(x)^2}$ etc., you should start with the substitution $u = f(x)$. We've already apply this principle in Example 7.3 and Exercise 7.9. This substitution is especially useful when $f(x) = \sqrt[n]{ax + b}$ is the root of some linear form.

Example 7.10. *Evaluate*

$$\int e^{\sqrt[3]{2x+5}} dx.$$

Solution. The function $f(x)$ from the above discussion is in our example $f(x) = \sqrt[3]{2x + 5}$. The substitution $u = f(x)$ yields $u^3 = 2x + 5$, and after differentiation $3u^2 du = 2dx$. We get

$$\int e^{\sqrt[3]{2x+5}} dx = \frac{3}{2} \int u^2 e^u du.$$

You now reduce the problem to the integration of a product between a polynomial and the exponential function. This you can attack using the strategy described above. $\square$

Example 7.11. *Evaluate*

$$\int \frac{1}{x + \sqrt{x + 2}} dx.$$

Solution. Our function is now $f(x) = \sqrt{x+2}$. The substitution $u = f(x)$ yields $u^2 = x + 2$, or $2udu = dx$. Therefore

$$\int \frac{1}{x+\sqrt{x+2}}dx = \int \frac{1}{u^2-2+u}2udu = \int \frac{2u}{(u-1)(u+2)}du.$$

To integrate this function you now use partial fractions: write

$$\frac{2u}{(u-1)(u+2)} = \frac{A}{u-1} + \frac{B}{u+2}$$

and solve for A, B. Plugging in $u = 1$ and $u = -2$ in the equality

$$2u = A(u+2) + B(u-1)$$

yields $A = 2/3$, $B = 4/3$. It follows that

$$\int \frac{2u}{(u-1)(u+2)}du = \frac{2}{3}\int \frac{1}{u-1}du + \frac{4}{3}\int \frac{1}{u+2}du = \frac{2}{3}\ln|u-1| + \frac{4}{3}\ln|u+2| + C.$$

□

In the next example, $f(x)$ is no longer of the form $\sqrt[n]{ax+b}$.

Example 7.12.

$$\int t^3 e^{-t^2}dt.$$

Solution. We let our f be $f(t) = t^2$. The substitution $y = f(t)$ yields $dy = 2t \cdot dt$ and

$$\int t^3 e^{-t^2}dt = \frac{1}{2}\int t^2 e^{-t^2}(2t \cdot dt) = \frac{1}{2}\int ye^{-y}dy.$$

Letting $u = y$, $dv = e^{-y}dy$, $v = -e^{-y}$ we get

$$\int ye^{-y}dy = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y} + C.$$

It follows that

$$\int t^3 e^{-t^2}dt = \frac{1}{2}(-t^2e^{-t^2} - e^{-t^2}) + C.$$

□

The logarithm. In many cases you are asked to integrate a function of the form $f(x)\ln(x)$. The first thing you should try is to use integration by parts with $u = \ln(x)$, $dv = f(x)dx$. This will require to integrate $f(x)$ in order to determine v, which will be in general an easy task.

Example 7.13. *Calculate*

$$\int \frac{\ln(x)}{x^2}dx.$$

Solution. The integrand has the form $f(x)\ln(x)$ where $f(x) = \frac{1}{x^2}$. We try integration by parts with $u = \ln(x)$, $dv = f(x)dx$, which gives $du = \frac{1}{x}dx$, $v = \frac{-1}{x}$. We get

$$\int \frac{\ln(x)}{x^2}dx = \int udv = uv - \int vdu = \frac{-\ln(x)}{x} + \int \frac{1}{x^2}dx = \frac{-\ln(x) - 1}{x} + C.$$

□

A more involved computation is presented in the following example.

Example 7.14. *Calculate*

$$\int \frac{x\ln(x)}{\sqrt{x^2-1}}dx.$$

Solution. The integrand has the form $f(x)\ln(x)$ where $f(x) = \frac{x}{\sqrt{x^2-1}}$. We try integration by parts with $u = \ln(x)$, $dv = f(x)dx$, $du = \frac{1}{x}dx$. The first step is to integrate f:

$$v = \int f(x)dx = \int \frac{x}{\sqrt{x^2-1}}dx.$$

Notice that the integrand involves the composition of the functions $\sqrt{}$ and $g(x)$, where $g(x) = x^2 - 1$. The preceding paragraph suggest the substitution $y = x^2 - 1$. We get $dy = 2xdx$ and

$$\int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{1}{\sqrt{y}}dy = \frac{1}{2}\sqrt{y} + C.$$

We can therefore take $v = \sqrt{x^2-1}$. Going back to integration by parts we get

$$\int \frac{x\ln(x)}{\sqrt{x^2-1}}dx = \int udv = uv - \int vdu = \ln(x)\sqrt{x^2-1} - \int \frac{\sqrt{x^2-1}}{x}dx.$$

To integrate $\frac{\sqrt{x^2-1}}{x}$ we can use one of the following methods

Method 1 Use the substitution $x = \sec(\theta)$. This gives $dx = \tan(\theta)\sec(\theta)d\theta$, $\sqrt{x^2-1} = \tan(\theta)$, and therefore

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x}dx &= \int \frac{\tan(\theta)}{\sec(\theta)}\tan(\theta)\sec(\theta)d\theta \\
&= \int \tan^2(\theta)d\theta = \int (\sec^2(\theta) - 1)d\theta \\
&= \tan(\theta) - \theta + C = \sqrt{x^2-1} - \sec^{-1}(x) + C.
\end{aligned}$$

Method 2 Use the substitution $y = \sqrt{x^2-1}$. This gives $y^2 = x^2 - 1$, and after differentiating $2ydy = 2xdx$. Therefore

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x}dx &= \int \frac{y}{x^2}xdx = \int \frac{y}{y^2+1}ydy \\
&= \int (1 - \frac{1}{y^2+1})dy = y - \tan^{-1}(y) + C = \sqrt{x^2-1} - \tan^{-1}(\sqrt{x^2-1}) + C.
\end{aligned}$$

Putting together all of the above, we get that

$$\int \frac{x\ln(x)}{\sqrt{x^2-1}}dx = \ln(x)\sqrt{x^2-1} - \sqrt{x^2-1} + \tan^{-1}(\sqrt{x^2-1}) + C.$$

□

The following is an instructive example of combining three of the strategies described above: use of rationalizing substitution $u = \sqrt[n]{ax+b}$, use of integration by parts to eliminate the logarithm, and use of partial fractions to integrate a rational function.

Example 7.15. *Calculate*

$$\int \frac{\ln(x-1)}{x\sqrt{x}}dx.$$

Solution. To get rid of the square root start with the substitution $t = \sqrt{x}$, or $x = t^2$. This gives $dx = 2tdt$ and

$$\int \frac{\ln(x-1)}{x\sqrt{x}}dx = \int \frac{\ln(t^2-1)}{t^3}2tdt = 2\int \frac{\ln(t^2-1)}{t^2}dt.$$

To eliminate the ln term, use integration by parts with $u = \ln(t^2-1)$, $dv = \frac{dt}{t^2}$. We get $du = \frac{2t}{t^2-1}dt$, $v = \frac{-1}{t}$, and hence

$$\begin{aligned}\int \frac{\ln(t^2-1)}{t^2}dt &= -\frac{\ln(t^2-1)}{t} + \int \frac{1}{t}\frac{2t}{t^2-1}dt \\ &= -\frac{\ln(t^2-1)}{t} + \int \frac{2}{(t+1)(t-1)}dt.\end{aligned}$$

Write

$$\frac{2}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1}$$

and solve for A, B to get $A = 1, B = -1$. It follows that

$$\int \frac{2}{(t+1)(t-1)}dt = \int \frac{1}{t-1}dt - \int \frac{1}{t+1}dt = \ln|t-1| - \ln|t+1| + C.$$

Tracing back through the calculations and using the fact that $t = \sqrt{x}$ we get

$$\int \frac{\ln(x-1)}{x\sqrt{x}}dx = -\frac{2\ln(x-1)}{\sqrt{x}} + 2\ln|\sqrt{x}-1| - 2\ln|\sqrt{x}+1| + C.$$

□

8 Applications

Arc length formula. The length of the curve $y = f(x)$, $a \le x \le b$ is

$$L = \int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

Example 8.1. *Determine the length of the curve* $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$, $0 \le x \le 1$.

Solution. We have
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x-x^2}}(x-x^2)' + \frac{1}{\sqrt{1-\sqrt{x}^2}}(\sqrt{x})' = \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} = \frac{2(1-x)}{2\sqrt{x-x^2}} = \frac{(1-x)}{\sqrt{x-x^2}}.$$
It follows that
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(1-x)^2}{x-x^2} = 1 + \frac{1-x}{x} = \frac{1}{x}.$$
Using the arc length formula we get
$$L = \int_0^1 \sqrt{\frac{1}{x}} dx = 2\sqrt{x}|_0^1 = 2.$$
□

Area of a surface of revolution. The surface area obtained by rotating the curve $y = f(x)$ about the x-axis is
$$S = \int 2\pi y ds.$$
Similarly, the surface area obtained by rotating the curve about the y-axis is
$$S = \int 2\pi x ds.$$
Example 8.2. *Determine the area of the surface of revolution obtained by rotating the curve* $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$, $1 \leq x \leq 2$ *about the y-axis.*

Solution. We have
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$$
It follows that
$$S = 2\pi \int x ds = 2\pi \int_1^2 x\left(\frac{x}{2} + \frac{1}{2x}\right) dx = 2\pi(\frac{x^3}{6} + \frac{x}{2})|_1^2 = \frac{10\pi}{3}.$$
□

Moments and centers of mass The coordinates of the center of mass are computed using the formulas
$$\overline{x} = \frac{M_y}{m},\ \overline{y} = \frac{M_x}{m},$$
where m is the mass of the system, and M_x, M_y are the moments with respect to the $x-$ and y-axes. In the case of a region lying between the axes $x = a$ and $y = b$, beneath the graph of a function $f(x)$ and above the graph of $g(x)$, the formulas for the mass and moments are (ρ denotes the density, which may be harmlessly taken to be equal to 1)
$$m = \rho \int_a^b (f(x) - g(x))dx,$$
$$M_x = \rho \int_a^b x(f(x) - g(x))dx,$$
$$M_y = \rho \int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)dx.$$

Example 8.3. *Find the centroid of the region bounded by the curves*

$$y = x^3, \;\; x + y = 2, \;\; y = 0.$$

First solution. Consider the two regions X, Y in the figure below. We denote by $M(X)_x, M(X)_y$ the moments of X with respect to the x- and y-axes, and similarly $M(Y)_x, M(Y)_y$ the corresponding moments of Y. We let $m(X), m(Y)$ denote the masses corresponding to X, Y (there is no loss in assuming the density to be equal to 1, so that $m(X), m(Y)$ are the areas of X, Y). We have

$$m(X) = \int_0^1 x^3 dx = \frac{1}{4}$$

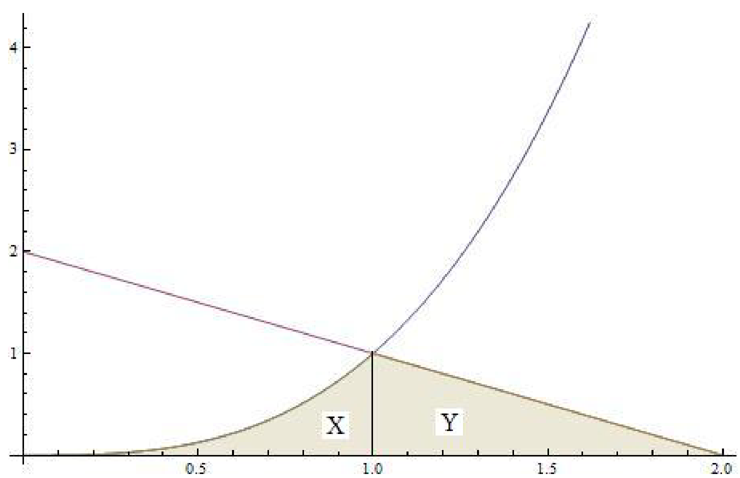
and

$$m(Y) = \frac{1}{2}.$$

(we used the fact that the area of a right triangle with side lengths $1, 1, \sqrt{2}$ is $1/2$). Using the formulas for moments we get

$$M(X)_y = \int_0^1 x \cdot x^3 dx = \frac{1}{5},$$

$$M(X)_x = \int_0^1 \frac{1}{2} \cdot (x^3)^2 dx = \frac{1}{14}.$$



We know that the coordinates of the centroid of a triangle are the averages of the coordinates of its vertices. Therefore $G(Y)$, the centroid of Y, has coordinates $\left(\frac{1+1+2}{3}, \frac{0+1+0}{3}\right) = (4/3, 1/3)$. This shows that

$$M(Y)_y = m(Y) \cdot \frac{4}{3} = \frac{2}{3},$$

$$M(Y)_X = m(Y) \cdot \frac{1}{3} = \frac{1}{6}.$$

It follows that the moments of the union $X \cup Y$ are

$$M_y = M(X)_y + M(Y)_y = \frac{13}{15},$$

$$M_x = M(X)_x + M(Y)_x = \frac{5}{21}.$$

The total mass of the system is

$$m = m(X) + m(Y) = \frac{3}{4}.$$

We conclude that the coordinates of the centroid of the system are

$$\overline{x} = \frac{M_y}{m} = \frac{52}{15},\ \overline{y} = \frac{M_x}{m} = \frac{20}{63}.$$

$\square$

Second solution. The region bounded by the curves $y = x^3, x + y = 2, y = 0$ is the region under the graph of f (see the figure), where

$$f(x) = \begin{cases} x^3, & 0 \le x \le 1; \\ 2 - x, & 1 \le x \le 2. \end{cases}$$

The coordinates $(\overline{x}, \overline{y})$ of the centroid are given by

$$\overline{x} = \frac{1}{A}\int_0^2 xf(x)dx,\ \overline{y} = \frac{1}{A}\int_0^2 \frac{1}{2}f(x)^2dx,$$

where A is the area below the graph of f,

$$A = \int_0^2 f(x)dx = \int_0^1 x^3dx + \int_1^2 (2-x)dx = \frac{x^4}{4}|_0^1 - \frac{(2-x)^2}{2}|_1^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

We have

$$\int_0^2 xf(x)dx = \int_0^1 x^4dx + \int_1^2 (2x - x^2)dx = \frac{x^5}{5}|_0^1 + (x^2 - \frac{x^3}{3})|_1^2 = \frac{1}{5} + \frac{4}{3} - \frac{2}{3} = \frac{13}{15},$$

and

$$\int_0^2 f(x)^2dx = \int_0^1 x^6dx + \int_1^2 (2-x)^2dx = \frac{x^7}{7}|_0^1 - \frac{(2-x)^3}{3}|_1^2 = \frac{1}{7} + \frac{1}{3} = \frac{10}{21}.$$

It follows that

$$\overline{x} = \frac{4}{3} \cdot \frac{13}{15} = \frac{52}{45}$$

and

$$\overline{y} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{10}{21} = \frac{20}{63}.$$

$\square$

Third solution. We can think of the given region as being bounded by the functions $f(y) = 2 - y$ and $g(y) = \sqrt[3]{y}$, $0 \le y \le 1$. Then the formulas for mass and moments give ($\rho = 1$)

$$m = \int_0^1 (f(y) - g(y))dy = (2y - \frac{y^2}{2} - \frac{3}{4}y^{4/3})|_0^1 = \frac{3}{4},$$

$$M_x = \int_0^1 y(f(y) - g(y))dy = (y^2 - \frac{y^3}{3} - \frac{3}{7}y^{7/3})|_0^1 = \frac{5}{21},$$

$$M_y = \int_0^1 \frac{1}{2}(f(y)^2 - g(y)^2)dy = \int_0^1 \frac{4 - 4y + y^2 - y^{2/3}}{2}dy = (2y - y^2 + \frac{y^3}{6} - \frac{3}{10}y^{5/3})|_0^1 = \frac{13}{15}.$$

We can now conclude as in the first solution. $\square$