Worksheet 19

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November 9, 2009

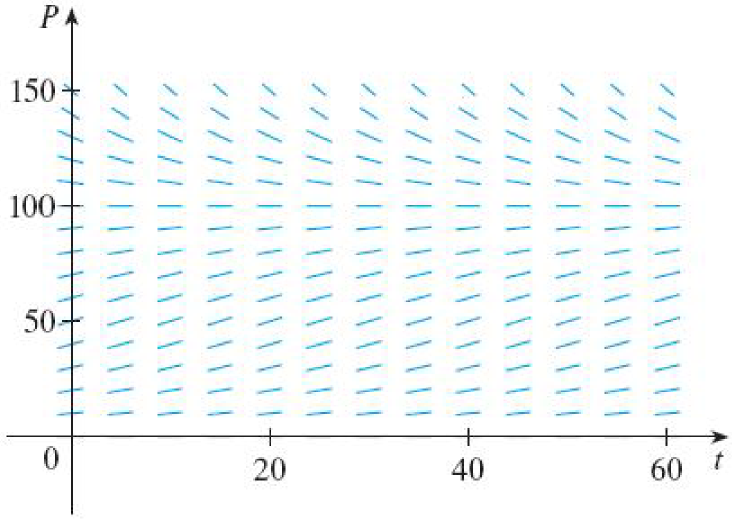
1. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?

(b) A direction field for this equation is shown. Where are the slopes close to zero? Where are they largest? Which solutions are increasing? Which solutions are decreasing?



(c) Use the direction field to sketch solutions for initial populations of $20, 40, 60, 80, 120$ and 140. What do these solutions have in common? How do they differ? Which solutions have inflection points? At which population levels do they occur?

(d) What are the equilibrium solutions? How are the other solutions related to these solutions?

2. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10000. The number of fish tripled in the first year.

(a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

(b) How long it will take for the population to increase to 5000.

3. Consider a population $P = P(t)$ with constant relative birth and death rates α and β, respectively, and a constant emigration rate m, where α, β, and m are positive constants. Assume that $\alpha > \beta$. Then the rate of change of the population at time t is modeled by the differential equation

$$\frac{dP}{dt} = kP - m, \text{ where } k = \alpha - \beta.$$

(a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.

(b) What condition on m will lead to an exponential expansion of the population?

(c) What condition on m will result in a constant population? A population decline?

(d) In 1847, the population of Ireland was about 8 million and the difference between the relative birth and death rates was 1.6% of the population. Because of the potato famine in the 1840s and 1850s, about 210000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time?

4. Let c be a positive number. A differential equation of the form

$$\frac{dy}{dy} = ky^{1+c}$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than the exponent 1 for natural growth.

(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.

(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t\to T^-} y(t) = \infty$.

(c) An especially prolific breed of rabbits has the growth term $ky^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

5. To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)dx}$ and integrate both sides. Solve the following linear differential equations

(a) $y' = x + 5y$.

(b) $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

(c) $x^2y' + xy = 1$, $x > 0$, $y(1) = 2$.

(d) $t^3 - t\dfrac{dy}{dt} = 2y$, $t > 0$, $y(1) = 0$.