Math 1B, Sections 108 and 110, Fall '09
Quiz 10, November 18

Show all work clearly and in order! You have 15 minutes to take this quiz.

1. (4 points) One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor. Write the differential equation that is satisfied by y and solve it.

   Solution. If y denotes the fraction of the population who have heard the rumor, then $1-y$ represents the fraction of the population who haven't heard the rumor. The rate of spread of the rumor ($y'(t)$) being proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor can then be rewritten as
   $$\frac{dy}{dt} = ky(1-y),$$
   for some positive constant k. This is a separable differential equation, so to solve it we separate the variables
   $$\frac{dy}{y(1-y)} = kdt$$
   and integrate
   $$\int \frac{dy}{y(1-y)} = \int kdt \Leftrightarrow \ln\left|\frac{y}{1-y}\right| = kt + c.$$
   Exponentiating, we get
   $$\left|\frac{y}{1-y}\right| = e^{kt+c}.$$
   Denoting e^c by A, and taking into account that $y \in (0,1)$ we get
   $$\frac{y}{1-y} = Ae^{kt} = \frac{Ae^{kt}}{1}.$$
   Adding the numerators to the denominators in the above equality of fractions we obtain
   $$y = \frac{y}{(1-y)+y} = \frac{Ae^{kt}}{1+Ae^{kt}}.$$
   $\square$

2. (3 points) Solve the initial-value problem
   $$(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0, \;\; y(0) = 1.$$

First solution. The differential equation is equivalent to

$$\frac{dy}{dx} = \frac{3x(1-y)}{x^2+1}.$$

This shows that $y' = 0$ whenever $y = 1$, that is the direction field is horizontal along the line $y = 1$. It follows that any solution $y(x)$ of the differential equation which takes the value 1 at some point x_0 has to be constantly equal to 1. Since the initial condition of our problem is $y(0) = 1$, the solution must be $y \equiv 1$. $\square$

Second solution. The differential equation is equivalent to

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{3x}{x^2+1},$$

which is a linear equation, with $P(x) = Q(x) = \dfrac{3x}{x^2+1}$. We have

$$\int P(x)dx = \frac{3}{2}\int \frac{2x}{x^2+1}dx = \frac{3}{2}\ln(x^2+1).$$

It follows that the integrating factor $I(x)$ is then

$$I(x) = e^{\int P(x)dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}.$$

The general technique for solving linear differential equations yields

$$yI(x) = \int Q(x)I(x)dx = \int \frac{3x}{x^2+1}(x^2+1)^{3/2}dx = \int 3x(x^2+1)^{1/2} = (x^2+1)^{3/2} + c.$$

Dividing by $I(x)$ we get

$$y = 1 + \frac{c}{(x^2+1)^{3/2}}.$$

The initial condition $y(0) = 1$ yields $1 = 1 + c$, i.e. $c = 0$. Therefore $y \equiv 1$. $\square$

Third solution. The differential equation is separable. We get by separating the variables

$$\frac{dy}{y-1} = \frac{-3x}{x^2+1}.$$

Integrating, we obtain

$$\ln|y-1| = \frac{-3}{2}\ln(x^2+1) + c,$$

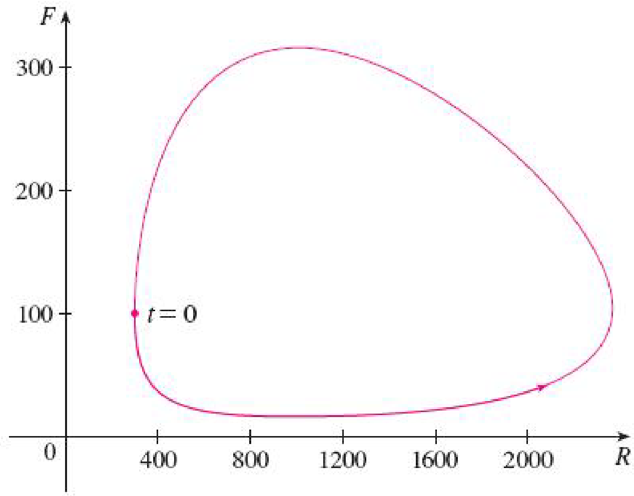
which yields by exponentiation

$$|y-1| = \frac{e^c}{(x^2+1)^{3/2}}.$$

Substituting e^c by a constant A, and allowing A to also be negative or 0, we get

$$y - 1 = \frac{A}{(x^2+1)^{3/2}}.$$

The initial condition $y(0) = 1$ implies that $1 - 1 = A/1 = A$, hence $A = 0$ and $y \equiv 1$. $\square$



3. (3 points) A phase trajectory is shown for populations of rabbits (R) and foxes (F).

   (a) Describe how each population changes as time goes by.

   (b) Use your description to make a rough sketch of the graphs of R and F as functions of time.

   Solution. See Example 1 in Stewart, p. 609, for a more detailed analysis of a similar predator-prey system. □