

Show all work clearly and in order! You have 20 minutes to take this quiz.

1. (5 points) Use the method of undetermined coefficients to solve the boundary value problem

$$y'' - y' = 1 + e^x, \quad y(0) = y(1) = 0.$$

*Solution.* The complementary equation is  $y'' - y' = 0$ . Its auxiliary equation is  $r^2 - r = 0$ , with roots  $r_1 = 0$ ,  $r_2 = 1$ . It follows that the general solution of the complementary equation has the form

$$y_c = c_1 e^{0 \cdot x} + c_2 e^{1 \cdot x} = c_1 + c_2 e^x.$$

We look for a particular solution  $y_p$  of the form  $y_{p_1} + y_{p_2}$ , where  $y_{p_1}$  is a particular solution of the equation

$$y'' - y' = 1.$$

and  $y_{p_2}$  is a particular solution of the equation

$$y'' - y' = e^x.$$

To find  $y_{p_1}$ , notice that 1 is a constant, so the general method of undetermined coefficients says that we should look for solutions of the form  $y_{p_1} = a$  with  $a$  a constant. But  $a$  is also a solution of the complementary equation, so we need to multiply it by  $x$ , i.e. take  $y_{p_1} = ax$ . We get

$$y'_{p_1} = a, \quad y''_{p_1} = 0,$$

hence  $y''_{p_1} - y'_{p_1} = -a = 1$ , or equivalently  $a = -1$ , yielding  $y_{p_1} = -x$ .

Similarly, we observe that we cannot choose  $y_{p_2}$  of the form  $be^x$  because it would be a solution of the complementary equation, so we must take  $y_{p_2} = bxe^x$ . We get

$$y'_{p_2} = b(e^x + xe^x) = b(x+1)e^x,$$

$$y''_{p_2} = b(e^x + (x+1)e^x) = b(x+2)e^x,$$

and therefore  $y''_{p_2} - y_{p_2} = be^x = e^x$ . It follows that  $b = 1$  and  $y_{p_2} = xe^x$ . The general solution of the given differential equation is therefore of the form

$$y = y_c + y_p = y_c + y_{p_1} + y_{p_2} = c_1 + c_2 e^x - x + xe^x.$$

The boundary conditions  $y(0) = y(1) = 0$  yield

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 + c_2 e - 1 + e = 0,$$

or equivalently  $c_2 = -c_1$  and  $c_1(1 - e) = 1 - e$ . We get  $c_1 = 1$ ,  $c_2 = -1$  and

$$y(x) = 1 - e^x - x + xe^x = (1 - x)(1 - e^x).$$

□

2. (5 points) Solve the differential equation using the method of variation of parameters.

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

*Solution.* The complementary equation is  $y'' - 2y' + y = 0$ . Its auxiliary equation is  $r^2 - 2r + 1 = 0$ , with roots  $r_1 = r_2 = 1$ . It follows that the general solution of the complementary equation has the form

$$y_c = c_1 y_1 + c_2 y_2,$$

where  $y_1 = e^x$ ,  $y_2 = x e^x$ . The general solution of the given differential equation can be written as

$$y = y_c + y_p$$

where  $y_p$  is a particular solution. Using the method of variation of parameters, we search for  $y_p$  of the form  $u_1 y_1 + u_2 y_2$ , with  $u_1, u_2$  functions whose derivatives satisfy the system of equations

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = F(x) = \frac{e^x}{1+x^2} \end{cases}$$

Substituting  $y_1, y_2$  in the above system we obtain

$$\begin{cases} u_1' e^x + u_2' x e^x = 0 \\ u_1' e^x + u_2' (x+1) e^x = \frac{e^x}{1+x^2} \end{cases}$$

Subtracting the first equation from the second, we get

$$u_2' e^x = \frac{e^x}{1+x^2} \Rightarrow u_2' = \frac{1}{1+x^2} \Rightarrow u_2 = \arctan(x).$$

Dividing by  $e^x$  in the first equation we get

$$u_1' = -u_2' x = -\frac{x}{1+x^2} \Rightarrow u_1 = \frac{-1}{2} \ln(1+x^2).$$

It follows that

$$y_p = \frac{-1}{2} \ln(1+x^2) e^x + \arctan(x) x e^x$$

and

$$y = y_c + y_p = e^x \left( c_1 + c_2 x - \frac{1}{2} \ln(1+x^2) + x \arctan(x) \right).$$

□

3. (3 points) Use power series to solve the initial value problem

$$y'' - xy' = x, \quad y(0) = 0, \quad y'(0) = 0.$$

(1 point) Give a closed formula for  $y'$  when  $y$  is the solution of the above problem.

*Solution.* We look for a solution  $y = \sum_{n=0}^{\infty} a_n x^n$ . The initial conditions  $y(0) = y'(0) = 0$  yield  $a_0 = a_1 = 0$ . We have

$$y' = \sum_{n \geq 0} n a_n x^{n-1}$$

hence

$$xy' = \sum_{n \geq 0} n a_n x^n.$$

Also,

$$y'' = \sum_{n \geq 0} n(n-1) a_n x^{n-2} = \sum_{n \geq 2} n(n-1) a_n x^{n-2} = \sum_{n \geq 0} (n+2)(n+1) a_{n+2} x^n.$$

We get

$$y'' - xy' = \sum_{n \geq 0} ((n+2)(n+1) a_{n+2} - n a_n) x^n = x.$$

The above equality is equivalent to

$$(n+2)(n+1) a_{n+2} = n a_n \text{ for } n \neq 1, \text{ and } 6a_3 - a_1 = 1.$$

We obtain the recursion relation

$$a_{n+2} = \frac{n}{(n+2)(n+1)} a_n, \quad n \geq 2,$$

and  $a_2 = 0$ ,  $a_3 = 1/6$ . It follows that  $a_{2n} = 0$  and for  $n \geq 1$

$$\begin{aligned} a_{2n+1} &= \frac{2n-1}{(2n+1)(2n)} a_{2n-1} = \frac{2n-1}{(2n+1)(2n)} \cdot \frac{2n-3}{(2n-1)(2n-2)} a_{2n-3} = \dots \\ &= \frac{2n-1}{(2n+1)(2n)} \cdot \frac{2n-3}{(2n-1)(2n-2)} \cdots a_3 \\ &= \frac{(2n-1)(2n-3) \cdots 3}{(2n+1)(2n)(2n-1)(2n-2) \cdots 5 \cdot 4} \cdot \frac{1}{6} \\ &= \frac{1}{2n+1} \cdot \frac{1}{2n(2n-2)(2n-4) \cdots 4 \cdot 2} \\ &= \frac{1}{2n+1} \cdot \frac{1}{2^n(n!)} \end{aligned}$$

We get

$$y = \sum_{n \geq 1} \frac{1}{2^n n!} \cdot \frac{x^{2n+1}}{2n+1}$$

and

$$y' = \sum_{n \geq 1} \frac{x^{2n}}{2^n n!} = \sum_{n \geq 1} \frac{(x^2/2)^n}{n!} = e^{x^2/2} - 1.$$

To get  $y'$  you can also make the substitution  $z = y'$ , and obtain that  $z$  is the solution of the linear 1st order differential equation

$$z' - xz = x,$$

which you can solve using the methods of chapter 9. □