Math 1B, Section 108, Fall '09
Quiz 1, September 2

Name: ______________________
SID: ______________________

Show all work clearly and in order! You have 15 minutes to take this quiz.

1. (3 points) First make a substitution and then use integration by parts to evaluate the integral
$$\int \cos(\sqrt{x})dx.$$

Solution. Make the substitution $y = \sqrt{x}$. We have $y^2 = x$ and $2y \cdot dy = dx$, thus
$$\int \cos(\sqrt{x})dx = 2\int y\cos(y)dy.$$
Use now integration by parts, with $u = y$, $dv = \cos(y)dy$, $v = \sin(y)$:
$$\int y\cos(y)dy = y\sin(y) - \int \sin(y)dy = y\sin(y) + \cos(y) + C.$$
It follows that
$$\int \cos(\sqrt{x})dx = 2(\sqrt{x}\sin(\sqrt{x}) + \cos(\sqrt{x})) + C.$$
□

2. (a) (2 points) Prove the reduction formula
$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx.$$

(b) (2 points) Evaluate $\int \cos^3(x)dx$.

Solution. (See also Stewart, Example 6, Section 7.1) (a) It suffices to prove that after taking derivatives we get an identity. We have
$$\left(\int \cos^n(x)dx\right)' = \cos^n(x),$$
$$\left(\frac{1}{n}\cos^{n-1}(x)\sin(x)\right)' = \frac{1}{n}((n-1)\cos^{n-2}(x)(-\sin^2(x)) + \cos^n(x)),$$
$$\left(\int \cos^{n-2}(x)dx\right)' = \cos^{n-2}(x).$$

It follows that we need to check the following identity

$$\cos^n(x) = \frac{1}{n}\left((n-1)\cos^{n-2}(x)(-\sin^2(x)) + \cos^n(x)\right) + \frac{n-1}{n}\cos^{n-2}(x),$$

which follows easily by replacing $-\sin^2(x)$ with $\cos^2(x) - 1$.

(b) Letting $n = 3$ in (a) we get

$$\int \cos^3(x)dx = \frac{1}{3}\cos^{3-1}(x)\sin(x) + \frac{3-1}{3}\int \cos^{3-2}(x)dx = \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\sin(x).$$

$\square$

3. (3 points) Evaluate

$$\int_0^1 \sqrt{1-x^2}dx.$$

(Hint: Let $u = \sqrt{1-x^2}$, $dv = dx$, and use integration by parts. Alternatively, make the substitution $x = \sin(y)$ and use the reduction formula 2(a).)

First solution. Letting $u = \sqrt{1-x^2}$, $dv = dx$, we have $du = \dfrac{-x}{\sqrt{1-x^2}}$ and $v = x$. Using integration by parts we get

$$\int_0^1 \sqrt{1-x^2}dx = x\sqrt{1-x^2}\,\Big|_0^1 - \int_0^1 \frac{-x^2}{\sqrt{1-x^2}}dx.$$

The function $x\sqrt{1-x^2}$ is zero when $x = 0, 1$, so we are left with evaluating

$$\begin{aligned}
-\int_0^1 \frac{-x^2}{\sqrt{1-x^2}}dx &= -\int_0^1 \frac{1-x^2}{\sqrt{1-x^2}}dx + \int_0^1 \frac{1}{\sqrt{1-x^2}}dx \\
&= -\int_0^1 \sqrt{1-x^2}dx + \arcsin(x)\,\Big|_0^1 \\
&= -\int_0^1 \sqrt{1-x^2}dx + \frac{\pi}{2}.
\end{aligned}$$

It follows that $2\displaystyle\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{2}$, or $\displaystyle\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$.

$\square$

Second solution. The substitution $x = \sin(y)$ yields $\sqrt{1-x^2} = \cos(y)$ and $dx = \cos(y)dy$. Therefore

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \cos^2(y)dy,$$

which, using the formula from 2(a) with $n = 2$, is equal to

$$\frac{1}{2}\cos(y)\sin(y)\,\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} 1dy = 0 + \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

$\square$