Math 1B, Sections 108 and 110, Fall '09
Quiz 2, September 9

Name: ______________________
SID: ______________________

Show all work clearly and in order! You have 15 minutes to take this quiz.

1. (3 points) Use your favorite method to evaluate

$$\int_0^1 \sqrt{1-x^2}dx.$$

Third solution. (First part is the same as in the 2nd solution) The substitution $x = \sin(y)$ yields $\sqrt{1-x^2} = \cos(y)$ and $dx = \cos(y)dy$. Therefore

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}dx &= \int_0^{\frac{\pi}{2}} \cos^2(y)dy \\ &= \int_0^{\frac{\pi}{2}} \frac{1+\cos(2y)}{2}dy \\ &= \left(\frac{y}{2} + \frac{\sin(2y)}{4}\right)\Bigg|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}.\end{aligned}$$

$\square$

Fourth solution. The arc of the unit circle contained in the first quadrant can be parameterized by $x \mapsto \sqrt{1-x^2}$, $x \in [0,1]$. It follows that the value of $\int_0^1 \sqrt{1-x^2}dx$ is precisely the area of the portion of the unit disk which is contained in the first quadrant. This is equal to one-fourth of the total area of the unit disk, i.e. $\frac{\pi}{4}$. $\square$

2. (3 points) Evaluate

$$\int \cos^2(x)\tan^3(x)dx.$$

Solution. The power of tangent is odd, so we make the substitution $u = \sec(x)$. We have

$$\int \cos^2(x)\tan^3(x)dx = \int \sec^{-3}(x)\tan^2(x)\sec(x)\tan(x)dx$$

and since $\tan^2(x) = u^2 - 1$, $du = \sec(x)\tan(x)dx$, we get

$$\begin{aligned}
\int \cos^2(x)\tan^3(x)dx &= \int \frac{u^2-1}{u^3}du \\
&= \ln|u| + \frac{1}{2u^2} + C \\
&= \ln|\sec(x)| + \frac{1}{2\sec^2(x)} + C \\
&= -\ln|\cos(x)| + \frac{1}{2}\cos^2(x) + C.
\end{aligned}$$

$\square$

3. (4 points) Evaluate the integral

$$\int \frac{1}{3\sin(x) - 4\cos(x)}dx.$$

(*Hint:* Use the substitution $t = \tan\left(\frac{x}{2}\right)$, the formulas

$$\sin(x) = \frac{2t}{1+t^2}, \ \cos(x) = \frac{1-t^2}{1+t^2},$$

and the fact that the polynomial $2t^2 + 3t - 2$ has roots -2 and $1/2$.)

Solution. The substitution $t = \tan\left(\frac{x}{2}\right)$ yields

$$\begin{aligned}
\int \frac{1}{3\sin(x) - 4\cos(x)}dx &= \int \frac{1}{3\cdot\frac{2t}{1+t^2} - 4\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} \\
&= \int \frac{1}{3t - 2 + 2t^2}dt \\
&= \frac{1}{2}\int \frac{1}{(t+2)(t-\frac{1}{2})}dt \\
&= \frac{1}{2}\int \frac{2}{5}\left(\frac{1}{t-\frac{1}{2}} - \frac{1}{t+2}\right)dt \\
&= \frac{1}{5}\ln\left(\frac{t-\frac{1}{2}}{t+2}\right) + C \\
&= \frac{1}{5}\ln\left(\frac{\tan\left(\frac{x}{2}\right) - \frac{1}{2}}{\tan\left(\frac{x}{2}\right) + 2}\right) + C.
\end{aligned}$$

$\square$