Math 1B, Section 108, Fall '09
Quiz 3, September 16

Name: ____________________
SID: ____________________

Show all work clearly and in order! You have 20 minutes to take this quiz.

1. (4 points) Calculate

$$\int \frac{\ln(x-1)}{x\sqrt{x}} dx.$$

(*Hint:* One way to do it is to start with a substitution, then use integration by parts, and finally integrate a rational function.)

Solution. To get rid of the square root start with the substitution $x = t^2$. This gives $dx = 2tdt$ and

$$\int \frac{\ln(x-1)}{x\sqrt{x}} dx = \int \frac{\ln(t^2-1)}{t^3} 2tdt = 2\int \frac{\ln(t^2-1)}{t^2} dt.$$

To eliminate the ln term, use integration by parts with $u = \ln(t^2-1)$, $dv = \frac{dt}{t^2}$. We get $du = \frac{2t}{t^2-1}dt$, $v = \frac{-1}{t}$, and hence

$$\begin{aligned}\int \frac{\ln(t^2-1)}{t^2} dt &= -\frac{\ln(t^2-1)}{t} + \int \frac{1}{t}\frac{2t}{t^2-1} dt \\ &= -\frac{\ln(t^2-1)}{t} + \int \frac{2}{(t+1)(t-1)} dt.\end{aligned}$$

Write

$$\frac{2}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1}$$

and solve for A, B to get $A = 1, B = -1$. It follows that

$$\int \frac{2}{(t+1)(t-1)} dt = \int \frac{1}{t-1} dt - \int \frac{1}{t+1} dt = \ln|t-1| - \ln|t+1| + C.$$

Tracing back through the calculations and using the fact that $t = \sqrt{x}$ we get

$$\int \frac{\ln(x-1)}{x\sqrt{x}} dx = -\frac{2\ln(x-1)}{\sqrt{x}} + 2\ln|\sqrt{x}-1| - 2\ln|\sqrt{x}+1| + C.$$

$\square$

2. (3 points) Determine whether the following integral is convergent or divergent. If convergent, evaluate it.

$$\int_{-\infty}^{\infty} x^5 e^{-x^6} dx.$$

Solution. We start by breaking up the integral into two parts

$$\int_0^\infty x^5 e^{-x^6} dx \text{ and } \int_{-\infty}^0 x^5 e^{-x^6} dx.$$

For the first integral, we have

$$\int_0^\infty x^5 e^{-x^6} dx = \lim_{a\to\infty} \int_0^a x^5 e^{-x^6} dx.$$

The substitution $u = x^6$ yields $du = 6x^5 dx$ and (pay attention to the limits of integration!)

$$\int_0^a x^5 e^{-x^6} dx = \frac{1}{6}\int_0^{a^6} e^{-u} du = \frac{-e^{-u}}{6}\Big|_0^{a^6} = \frac{1-e^{-a^6}}{6}.$$

Therefore

$$\lim_{a\to\infty} \int_0^a x^5 e^{-x^6} dx = \lim_{a\to\infty} \frac{1-e^{-a^6}}{6} = \frac{1}{6},$$

because $-a^6 \to -\infty$ when $a \to \infty$. Similarly

$$\int_{-\infty}^0 x^5 e^{-x^6} dx = \lim_{b\to-\infty} \int_b^0 x^5 e^{-x^6} dx.$$

The substitution $u = x^6$ yields $du = 6x^5 dx$ and (pay attention to the limits of integration!)

$$\int_b^0 x^5 e^{-x^6} dx = \frac{1}{6}\int_{b^6}^0 e^{-u} du = \frac{-e^{-u}}{6}\Big|_{b^6}^0 = \frac{-1+e^{-b^6}}{6}.$$

Therefore

$$\lim_{b\to-\infty} \int_b^0 x^5 e^{-x^6} dx = \lim_{b\to-\infty} \frac{-1+e^{-b^6}}{6} = \frac{-1}{6},$$

because $-b^6 \to -\infty$ when $b \to -\infty$.

It follows that the integral is convergent, and its value is equal to $\dfrac{1}{6} + \dfrac{-1}{6} = 0.$ □

3. (a) (1 point) Write down the general formula for approximating $\int_a^b f(x)dx$ using Simpson's rule. What is the condition on n (the number of intervals)?

   (b) (2 points) Compute the 10^{th} approximation S_{10} of

$$\int_0^{10} (4x^3 + 3x^2 + 2x + 1)dx.$$

   (*Hint:* You may use the fact that the error bound for Simpson's rule is given by

$$|E_S| \le \frac{K(b-a)^5}{180n^4},$$

   where K is any real number with the property that $|f^{(4)}(x)| \le K$ for $a \le x \le b$.)

Solution. (a) See Stewart, Section 7.7, pg. 502.

(b) $f(x) = 4x^3 + 3x^2 + 2x + 1$ is a polynomial of degree 3, hence $f^{(4)}(x) = 0$ for all x. It follows that we can take $K = 0$ in the error bound formula, and therefore $E_S = 0$. This says that S_{10}, the value given by the approximation formula, coincides with the actual value of the integral (notice that this assertion is independent of the number of intervals used for approximation). We get

$$S_{10} = \int_0^{10} (4x^3 + 3x^2 + 2x + 1)dx = (x^4 + x^3 + x^2 + x)|_0^{10} = 11110.$$

$\square$