Show all work clearly and in order! You have 15 minutes to take this quiz.

1. (3 points) Find the arc length function for the curve

\[ y = \sin^{-1}(x) + \sqrt{1 - x^2} \]

with starting point (0, 1).

**Solution.** Let \( s \) denote the arc length function. We have

\[ s(x) = \int_0^x \sqrt{1 + (y'(t))^2} \, dt. \]

Since

\[ y'(t) = \frac{1}{\sqrt{1 - t^2}} - \frac{t}{\sqrt{1 - t^2}}, \]

we get

\[ \sqrt{1 + (y'(t))^2} = \sqrt{1 + \left(\frac{1 - t}{1 - t^2}\right)^2} = \sqrt{1 + \frac{1 - t}{1 + t}} = \sqrt{\frac{2}{1 + t}}. \]

It follows that

\[ s(x) = \int_0^x \sqrt{2(1 + t)^{-1/2}} \, dt = \sqrt{2(2(1 + t)^{1/2})}\big|_0^x = 2\sqrt{2(\sqrt{1 + x} - 1)}. \]

\[ \square \]

2. (3 points) The curve

\[ y = 1 - x^2, \quad 0 \leq x \leq 1 \]

is rotated about the \( y \)-axis. Find the area of the resulting surface.

**Solution.** For rotation about the \( y \)-axis, the surface area formula is

\[ S = \int 2\pi x ds = 2\pi \int_0^1 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \]

\[ = 2\pi \int_0^1 x \sqrt{1 + 4x^2} \, dx. \]

The substitution \( u = 4x^2 \) yields \( du = 8xdx \), hence

\[ S = \frac{2\pi}{8} \int_0^1 \sqrt{1 + u} \, du = \frac{\pi}{4} \cdot \frac{2}{3} \left(1 + u\right)^{3/2}\big|_0^4 = \frac{\pi}{6} (5^{3/2} - 1). \]

\[ \square \]
3. (4 points) Find the centroid of the region bounded by the curves

\[ y = x^3, \quad x + y = 2, \quad y = 0. \]

**Solution.** The region bounded by the curves \( y = x^3, x + y = 2, y = 0 \) is the region under the graph of \( f \) (see the figure below), where

\[
f(x) = \begin{cases} x^3, & 0 \leq x \leq 1; \\ 2 - x, & 1 \leq x \leq 2. \end{cases}
\]

The coordinates \((\bar{x}, \bar{y})\) of the centroid are given by

\[
\bar{x} = \frac{1}{A} \int_0^2 xf(x) \, dx, \quad \bar{y} = \frac{1}{A} \int_0^2 \frac{1}{2} f(x)^2 \, dx,
\]

where \(A\) is the area below the graph of \(f\),

\[
A = \int_0^2 f(x) \, dx = \int_0^1 x^3 \, dx + \int_1^2 (2 - x) \, dx = \frac{x^4}{4} \bigg|_0^1 - \frac{(2-x)^2}{2} \bigg|_1^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.
\]

We have

\[
\int_0^2 xf(x) \, dx = \int_0^1 x^4 \, dx + \int_1^2 (2x - x^2) \, dx = \frac{x^5}{5} \bigg|_0^1 + (x^2 - \frac{x^3}{3}) \bigg|_1^2 = \frac{1}{5} + \frac{4}{3} - \frac{2}{3} = \frac{13}{15},
\]

and

\[
\int_0^2 f(x)^2 \, dx = \int_0^1 x^6 \, dx + \int_1^2 (2 - x)^2 \, dx = \frac{x^7}{7} \bigg|_0^1 - \frac{(2-x)^3}{3} \bigg|_1^2 = \frac{1}{7} + \frac{1}{3} = \frac{10}{21}.
\]

It follows that

\[
\bar{x} = \frac{4}{3} \cdot \frac{13}{15} = \frac{52}{45}
\]

and

\[
\bar{y} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{10}{21} = \frac{20}{63}.
\]