Math 1B, Sections 108 and 110, Fall '09
Quiz 4, September 23

Name: ________________________

SID: ________________________

Show all work clearly and in order! You have 15 minutes to take this quiz.

1. (3 points) Find the arc length function for the curve

$$y = \sin^{-1}(x) + \sqrt{1-x^2}$$

with starting point $(0, 1)$.

Solution. Let s denote the arc length function. We have

$$s(x) = \int_0^x \sqrt{1 + (y'(t))^2}dt.$$

Since

$$y'(t) = \frac{1}{\sqrt{1-t^2}} - \frac{t}{\sqrt{1-t^2}},$$

we get

$$\sqrt{1+(y'(t))^2} = \sqrt{1 + \frac{(1-t)^2}{1-t^2}} = \sqrt{1+\frac{1-t}{1+t}} = \sqrt{\frac{2}{1+t}}.$$

It follows that

$$s(x) = \int_0^x \sqrt{2}(1+t)^{-1/2}dt = \sqrt{2}(2(1+t)^{1/2})|_0^x = 2\sqrt{2}(\sqrt{1+x}-1).$$

$\square$

2. (3 points) The curve

$$y = 1 - x^2,\ 0 \le x \le 1$$

is rotated about the y-axis. Find the area of the resulting surface.

Solution. For rotation about the y-axis, the surface area formula is

$$\begin{aligned} S = \int 2\pi x ds &= 2\pi \int_0^1 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\ &= 2\pi \int_0^1 x\sqrt{1+4x^2}dx. \end{aligned}$$

The substitution $u = 4x^2$ yields $du = 8xdx$, hence

$$S = \frac{2\pi}{8}\int_0^4 \sqrt{1+u}du = \frac{\pi}{4}\cdot\frac{2}{3}(1+u)^{3/2}|_0^4 = \frac{\pi}{6}(5^{3/2}-1).$$

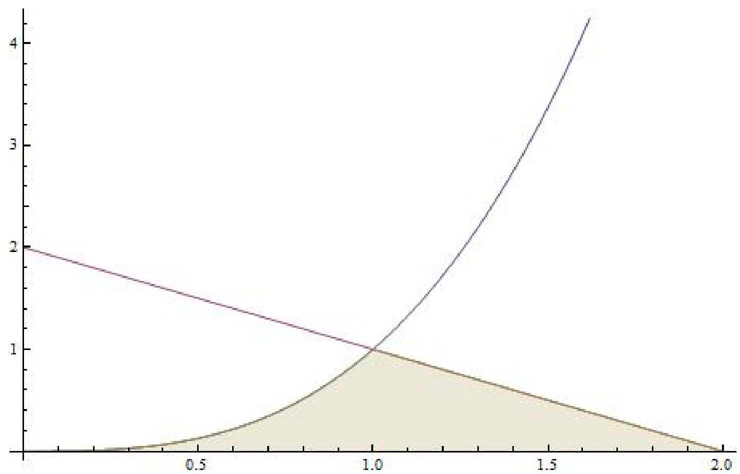
$\square$

3. (4 points) Find the centroid of the region bounded by the curves

$$y = x^3, \; x + y = 2, \; y = 0.$$

Solution. The region bounded by the curves $y = x^3, x + y = 2, y = 0$ is the region under the graph of f (see the figure below), where

$$f(x) = \begin{cases} x^3, & 0 \le x \le 1; \\ 2 - x, & 1 \le x \le 2. \end{cases}$$



The coordinates $(\overline{x}, \overline{y})$ of the centroid are given by

$$\overline{x} = \frac{1}{A}\int_0^2 x f(x)dx, \; \overline{y} = \frac{1}{A}\int_0^2 \frac{1}{2} f(x)^2 dx,$$

where A is the area below the graph of f,

$$A = \int_0^2 f(x)dx = \int_0^1 x^3 dx + \int_1^2 (2-x)dx = \frac{x^4}{4}|_0^1 - \frac{(2-x)^2}{2}|_1^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

We have

$$\int_0^2 x f(x)dx = \int_0^1 x^4 dx + \int_1^2 (2x - x^2)dx = \frac{x^5}{5}|_0^1 + (x^2 - \frac{x^3}{3})|_1^2 = \frac{1}{5} + \frac{4}{3} - \frac{2}{3} = \frac{13}{15},$$

and

$$\int_0^2 f(x)^2 dx = \int_0^1 x^6 dx + \int_1^2 (2-x)^2 dx = \frac{x^7}{7}|_0^1 - \frac{(2-x)^3}{3}|_1^2 = \frac{1}{7} + \frac{1}{3} = \frac{10}{21}.$$

It follows that

$$\overline{x} = \frac{4}{3} \cdot \frac{13}{15} = \frac{52}{45}$$

and

$$\overline{y} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{10}{21} = \frac{20}{63}.$$

$\square$