Math 1B, Sections 108 and 110, Fall '09
Quiz 6, October 14

Name: ______________________
SID: ______________________

Show all work clearly and in order! You have 15 minutes to take this quiz.

1. (3 points) Test for convergence or divergence the series
$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}.$$

Solution. We will use the Ratio Test to prove the convergence of the series. If we let $a_n = \frac{n!}{e^{n^2}}$, then we get
$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{e^{(n+1)^2}}}{\frac{n!}{e^{n^2}}} = \frac{n+1}{e^{(n+1)^2-n^2}} = \frac{n+1}{e^{2n+1}},$$
hence
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1.$$
According to the Ratio Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ is convergent. □

2. (3 points) Determine whether the series
$$\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$$
is convergent or divergent.

Solution. Let $a_n = (\sqrt[n]{n} - 1)^n$. We test the convergence of the series using the Root Test. We have
$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (\sqrt[n]{n} - 1) = 1 - 1 = 0 < 1.$$
According to the Root Test, the series $\sum a_n = \sum (\sqrt[n]{n} - 1)^n$ is convergent.
(To prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ it suffices to show that
$$\ln(\lim_{n\to\infty} \sqrt[n]{n}) = 0.$$
But ln is a continuous function, hence
$$\ln(\lim_{n\to\infty} \sqrt[n]{n}) = \lim_{n\to\infty} \ln(n^{1/n}) = \lim_{n\to\infty} \frac{1}{n} \ln\ n = \lim_{x\to\infty} \frac{\ln\ x}{x} \overset{\text{l'Hôpital}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0,$$
which is what we wanted to show.) □

3. (4 points) Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln\ n}{n}$$

is absolutely convergent, conditionally convergent, or divergent.

Solution. Let $a_n = (-1)^{n-1}\dfrac{\ln\ n}{n}$. We have

$$|a_n| = \frac{\ln\ n}{n} > \frac{1}{n}.$$

Since the harmonic series is divergent, it follows from the Comparison Test that $\sum |a_n|$ is also divergent, i.e. the series $\sum a_n$ is not absolutely convergent.

On the other hand, if we let $b_n = \dfrac{\ln\ n}{n}$, then $\sum a_n = \sum(-1)^{n-1}b_n$, and we can use the Alternating Series Test to check that $\sum a_n$ is convergent. In order to be able to apply the Alternating Series Test, we need to check that the sequence $\{b_n\}$ is eventually decreasing, and that $\lim\limits_{n\to\infty} b_n = 0$. The last part is clear, and to prove the first part it suffices to check that the function $f(x) = \dfrac{\ln\ x}{x}$ is eventually decreasing. We have

$$f'(x) = \frac{(\ln\ x)' \cdot x - \ln\ x \cdot x'}{x^2} = \frac{\frac{1}{x} \cdot x - \ln\ x}{x^2} = \frac{1 - \ln\ x}{x^2}.$$

Therefore as soon as $1 - \ln\ x < 0$ (or equivalently $x > e$), $f'(x)$ is negative and hence f is decreasing. This shows that $\{b_n\}$ is decreasing starting with $n = 3$, thus by the Alternating Series Test $\sum a_n = \sum(-1)^{n-1}b_n$ is convergent.

In conclusion, $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln\ n}{n}$ is convergent, but not absolutely convergent, i.e. it is conditionally convergent. $\square$