Math 1B, Section 110, Fall '09
Quiz 7, October 21

Name: ______________________
SID: ______________________

Show all work clearly and in order! You have 15 minutes to take this quiz.

1. (4 points) Find the radius of convergence and the interval of convergence of the series
$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{n^2+1}.$$

Solution. We let
$$a_n = \frac{n(x+1)^n}{n^2+1}$$
and use the ratio test to determine the radius of convergence. We have
$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)(x+1)^{n+1}}{(n+1)^2+1}}{\dfrac{n(x+1)^n}{n^2+1}} = \frac{n+1}{n} \cdot \frac{n^2+1}{n^2+2n+2} \cdot (x+1).$$
Since $\lim_{n\to\infty} \frac{n+1}{n} = 1$ and $\lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} = 1$ we get
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x+1|.$$
According to the ratio test, convergence of the series holds for $|x+1| < 1$ (which is the same as $|x-(-1)| < 1$). It follows that the radius of convergence is equal to 1, and the interval of convergence contains the interval $(-1-1, -1+1) = (-2, 0)$. In order to compute the interval of convergence, we only need to test whether the series is convergent or divergent when $x = -2, 0$.

If $x = -2$ we get
$$\sum_{n=1}^{\infty} \frac{n(-2+1)^n}{n^2+1} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1},$$
which is convergent by the Alternating Series Test.

If $x = 0$ we get
$$\sum_{n=1}^{\infty} \frac{n(0+1)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{n}{n^2+1}.$$
This behaves the same as $\sum \frac{n}{n^2} = \sum \frac{1}{n}$ (Limit Comparison Test), which is divergent by the p-series test.

In conclusion, the radius of convergence is $R = 1$ and the interval of convergence is $I = [-2, 0)$. $\square$

2. (3 points) Find the power series representation for

$$f(x) = \ln(5-x).$$

Solution. We have

$$f'(x) = -\frac{1}{5-x} = \frac{-1}{5} \cdot \frac{1}{1-\frac{x}{5}} = \frac{-1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{x^n}{5^{n+1}}\right)$$

It follows that

$$f(x) = \int f'(x)dx = \sum_{n=0}^{\infty}\left(-\frac{x^{n+1}}{5^{n+1}(n+1)}\right) + C.$$

To determine C, we plug in $x=0$ and get $f(0) = 0 + C$, i.e. $C = \ln\ 5$. □

3. (3 points) Evaluate the indefinite integral

$$\int \frac{2t^2}{1-t^7}dt$$

as a power series. What is the radius of convergence?

Solution. We have

$$\int \frac{2t^2}{1-t^7}dt = \int 2t^2\left(\sum_{n=0}^{\infty} t^{7n}\right)dt = \int\left(\sum_{n=0}^{\infty} 2t^{7n+2}\right)dt = \sum_{n=0}^{\infty}\frac{2t^{7n+3}}{7n+3} + C.$$

To determine the radius of convergence, we have two methods:

<u>*Method 1.*</u> Use the fact that the radius of convergence for a power series is equal to the one for its integral, and note that $\sum_{n=0}^{\infty} 2t^{7n+2}$ is a geometric series, so has radius of convergence equal to 1.

<u>*Method 2.*</u> Use the ratio test for the sequence $a_n = \dfrac{2t^{7n+3}}{7n+3}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2t^{7(n+1)+3}}{7(n+1)+3}}{\dfrac{2t^{7n+3}}{7n+3}} = t^7 \cdot \frac{7n+3}{7n+10}.$$

Therefore

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |t^7|,$$

and the radius of convergence is determined by the inequality $|t^7| < 1 \iff |t| < 1$, i.e. it is equal to 1.

□