11.1 $\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty}\sqrt[n]{n+1} = \lim_{n\to\infty}\sqrt[n]{n-1} = \lim_{n\to\infty}\sqrt[n]{\ln n} = 1$, $\lim_{n\to\infty}\sqrt[n]{P(n)} = 1$ when P is a polynomial.

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ | $\lim_{n\to\infty} 2^{1/n} = 1 = \lim_{n\to\infty} c^{1/n}$ where $c>0$ is a constant | $\lim_{n\to\infty}\frac{2^{1/n}-1}{1/n} = \lim_{x\to 0}\frac{2^x-1}{x} \overset{H}{=} \lim_{x\to 0}\frac{2^x \ln 2}{1} = \ln 2$; $\lim n(c^{1/n}-1) = \ln c$ for $c>0$.

11.8 ROC=?, IOC=? Use ROOT/RATIO to get $|x-a|<R$. Test endpoints: $|x-a|=R \iff x=a\pm R$

e.g. $\sum_{n=1}^{\infty}\frac{x^{3n}}{n\cdot 3^n}$: ROOT $\sqrt[n]{\left|\frac{x^{3n}}{n\cdot 3^n}\right|} = \frac{|x|^3}{\sqrt[n]{n}\cdot 3} \xrightarrow{n\to\infty} \frac{|x|^3}{3} < 1 \iff |x|^3 < 3 \iff |x| < 3^{1/3}$. ROC $=3^{1/3}$, test $x=\pm 3^{1/3}$

$\underline{x=3^{1/3}}$: $\sum_{n=1}^{\infty}\frac{1}{n}$ DIV. $\underline{x=-3^{1/3}}$: $\sum_{n=1}^{\infty}\frac{(-1)^{3n}}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ CONV by AST. IOC $= [-3^{1/3}, 3^{1/3})$.

$\sum_{n=1}^{\infty}\frac{n(x-4)^n}{n^3+1}$: RATIO $\left|\frac{(n+1)(x-4)^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n(x-4)^n}\right| = \frac{n+1}{n}\cdot\frac{n^3+1}{(n+1)^3+1}\cdot|x-4| \to |x-4| < 1$ $a=4$, ROC $=R=1$ Endpoints $a\pm R = 4\pm 1 = 3, 5$

$\underline{x=3}$: $\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^3+1}$ CONV BY AST. $\underline{x=5}$ $\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ LCT to $b_n=\frac{1}{n^2}$ $\frac{\frac{n}{n^3+1}}{1/n^2} = \frac{n^3}{n^3+1}\to 1$. Since $\sum\frac{1}{n^2}$ Conv. $\Rightarrow \sum\frac{n}{n^3+1}$ Conv.

IOC $=[3,5]$

11.9,10 $\frac{1}{1-x} = 1+x+\dots = \sum_{n=0}^{\infty}x^n$ $R=1$

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ $R=\infty$

$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $R=\infty$

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}\dots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ $R=\infty$

$(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\dots$ $R=1$

$\tan^{-1}x = \int\frac{dx}{1+x^2} = \int(1-x^2+x^4-x^6+\dots)dx = \int\sum_{n=0}^{\infty}(-1)^n x^{2n}dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + C$

Plug in $x=0$ to get $C=0$. ROC $=1$ (same as for $\frac{1}{1+x^2}$)

$\ln(1-x) = \int\frac{-dx}{1-x} = -\int\sum_{n=0}^{\infty}x^n dx = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = -x-\frac{x^2}{2}-\frac{x^3}{3}-\dots$ ROC $=1$ (same as for $\frac{1}{1-x}$) | Again, there should be a C, but can check $C=0=\ln 1$

$\arcsin(x) = \int\frac{dx}{\sqrt{1-x^2}} = \int(1-x^2)^{-1/2}dx = \int\sum_{n=0}^{\infty}\binom{-1/2}{n}(-x^2)^n dx = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n\frac{x^{2n+1}}{2n+1}$ $+C$ ($C=\arcsin 0=0$). Since $\binom{-1/2}{n} = \frac{(-1)^n}{2^n}\cdot\frac{1\cdot 3\cdot\dots\cdot(2n-1)}{n!}$, get $\arcsin x = \sum_{n=0}^{\infty}\frac{1\cdot 3\cdot\dots\cdot(2n-1)}{2^n\cdot n!}\cdot\frac{x^{2n+1}}{2n+1}$ ROC $=1$ (as for $(1-x^2)^{-1/2}$)

ROC doesn't change when taking derivatives/integrate

$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ | $\binom{-1}{n} = \frac{(-1)(-2)\cdots(-n)}{n!} = \frac{(-1)^n\cdot n!}{n!} = (-1)^n$

$\binom{-2}{n} = \frac{(-2)(-3)\cdots(-(n+1))}{n!} = (-1)^n\cdot(n+1)$ | $\binom{-3}{n} = (-1)^n\frac{(n+1)(n+2)}{2}$

$\binom{-k}{n} = (-1)^n\binom{n+k-1}{n} = (-1)^n\binom{n+k-1}{k-1}$ when k is a natural number

$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2}\left(1-\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}\right) = \frac{1}{2}\left(\cancel{1}-\cancel{1}+\frac{(2x)^2}{2!}-\frac{(2x)^4}{4!}\dots\right)$

$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x)^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n-1}}{(2n)!}x^{2n}$ ROC $=\infty$ (as for $\cos 2x$)

$\binom{-1/2}{n} = \frac{(-1/2)(-3/2)\cdots\left(-\frac{2n-1}{2}\right)}{n!} = \frac{(-1)^n}{2^n}\cdot\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{n!}$

$\frac{x}{\sqrt{4+x^2}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n+1}}{2^{2n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot\dots\cdot(2n-1)}{2^{3n+1}\cdot n!}x^{2n+1}$

Taylor Series: $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

Maclaurin: $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot x^n$

Taylor polynomial: $T_n(x) = \sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$

$= f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n$

Remainder: $R_n(x) = f(x)-T_n(x)$.

Taylor's Inequality: If $|f^{(n+1)}(x)|\le M$ for $x\in I$ (I some interval, e.g. $I=[a-d,a+d]=\{x \mid |x-a|\le d\}$) then $|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}$ for $x\in I$.

Estimate $x\ln x$ at $a=1$, using the 3rd Taylor polynomial $T_3(x)$ on the interval $I=[0.5,1.5]$.

i	$f^{(i)}(x)$	$f^{(i)}(1)$
1	$\ln x+1$	1
2	$1/x$	1
3	$-1/x^2$	-1
4	$2/x^3$	DON'T CARE
0	$x\ln x$	0

$T_3(x) = f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3$

$= 0+\frac{1}{1!}(x-1)+\frac{1}{2!}(x-1)^2-\frac{1}{3!}(x-1)^3$

$= (x-1)+\frac{(x-1)^2}{2}-\frac{(x-1)^3}{6}$

$|f(x)-T_3(x)|\le\frac{M}{4!}|x-1|^4$, $M\ge|f^{(4)}(x)|$ for $0.5\le x\le 1.5$

$|f^{(4)}(x)| = 2/x^3$ which is decreasing on $[0.5,1.5]$, so its maximal value is attained at the left endpoint of the interval, i.e. at $x=0.5$. So $|f^{(4)}(x)|\le 2/0.5^3 = 16$. Take $M=16$

$|f(x)-T_3(x)|\le\frac{16}{24}|x-1|^4$

$-0.5\le x-1\le 0.5$, so $|x-1|\le 0.5$ So $|f(x)-T_3(x)|\le\frac{16}{24}\cdot 0.5^4 = \frac{1}{24}$

11.1 $\lim_{n\to\infty} n^a = \begin{cases}\infty, & a>0\\ 1, & a=0\\ 0, & a<0\end{cases}$ | $\lim_{n\to\infty}\frac{an^p+\dots}{bn^q+\dots} = \begin{cases}\infty, & p>q\\ a/b, & p=q\\ 0, & p<q\end{cases}$ | $\lim_{n\to\infty} r^n = \begin{cases}0, & -1<r<1\\ 1, & r=1\\ \infty, & r>1\\ \text{DNE}, & r\le -1\end{cases}$ | Every bounded monotonic sequence is convergent

11.2 $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = a+ar+\dots = \begin{cases}\frac{a}{1-r}, & |r|<1\\ \text{DNE}, & |r|\ge 1\end{cases}$

e.g. $\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$ DIV bc $\frac{4}{3}>1$

$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty}\frac{2}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1-1/3} + \frac{2/3}{1-2/3} = \frac{5}{2}$

Telescoping: $\sum_{n=2}^{\infty}\frac{1}{n(n+2)} = \sum_{n=2}^{\infty}\left(\frac{1/2}{n} - \frac{1/2}{n+2}\right) = \frac{1}{2}\left(\frac{1}{2} - \cancel{\frac{1}{4}} + \frac{1}{3} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{6}} - \frac{1}{8} + \cancel{\frac{1}{7}} - \frac{1}{9} + \cancel{\frac{1}{8}} - \frac{1}{10} \cdots\right) = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) = \frac{5}{12}$

$\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty}\left(\ln(n+1)-\ln n\right) = \lim_{n\to\infty}\left(\ln(n+1)-\ln 1\right) = \infty$

$\sum_{n=1}^{\infty}\left[f(n)-f(n+a)\right] = f(1)+\dots+f(a) - a\cdot\lim_{n\to\infty}\left[f(n)\right]$

TEST FOR DIVERGENCE: If $\lim_{n\to\infty} a_n \neq 0$ or DNE, then $\sum_{n=1}^{\infty} a_n$ is DIV

e.g. $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n}{n+1}$

11.3 INTEGRAL TEST: f continuous, positive, decreasing (eventually) $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)\,dx$ is convergent. e.g.: $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is $\begin{cases}\text{conv.}, & p>1\\ \text{div.}, & p\le 1\end{cases}$, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ is $\begin{cases}\text{conv.}, & p>1\\ \text{div.}, & p\le 1\end{cases}$, $\sum_{n=1}^{\infty}\frac{\ln n}{n^p}$ is $\begin{cases}\text{conv}, & p>1\\ \text{div.}, & p\le 1\end{cases}$

REMAINDER ESTIMATE: $R_n = s - s_n = a_{n+1}+a_{n+2}+\dots$, $s_n = a_1+\dots+a_n$; Then $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$.

11.4 COMPARISON TEST: $a_n, b_n \ge 0$. i) Assume $a_n \le b_n\ \forall n$. If $\sum_n a_n$ divergent, then $\sum_n b_n$ div. If $\sum_n b_n$ conv., then $\sum_n a_n$ conv.

e.g. $\sum_n\frac{\ln n}{n} \ge \sum\frac{1}{n} \Rightarrow \sum_n\frac{\ln n}{n}$ DIV. ii) Assume $a_n \ge b_n$. If $\sum_n a_n$ conv., then $\sum_n b_n$ conv. If $\sum_n b_n$ div, then $\sum a_n$ div.

LCT: $a_n, b_n > 0$, $\lim_{n\to\infty}\frac{a_n}{b_n} = c$ i) $c>0$ and finite → then either both series converge or both diverge.

ii) $c=0$: if $\sum_n b_n$ converges then $\sum_n a_n$ converges; if $\sum_n a_n$ DIV, then $\sum_n b_n$ DIV.

iii) $c=\infty$: if $\sum_n b_n$ diverges then $\sum_n a_n$ diverges; if $\sum_n a_n$ CONV, then $\sum_n b_n$ CONV.

e.g. $a_n = \sin\frac{1}{n}$, $b_n = \frac{1}{n}$, $\lim_{n\to\infty}\frac{a_n}{b_n} = 1 \Rightarrow \sum_n \sin\frac{1}{n}$ Div.

$a_n = \tan\left(\frac{1}{n^2+2n+3}\right)$, $b_n = \frac{1}{n^2+2n+3}$, $\lim_{n\to\infty}\frac{a_n}{b_n} = 1 \Rightarrow \sum_n \tan\left(\frac{1}{n^2+2n+3}\right)$ Conv.

$a_n = \frac{1}{n}$, $b_n = \frac{1}{n^{1+1/n}}$, $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty}\sqrt[n]{n} = 1 \Rightarrow \sum_n b_n$ DIV

REM. ESTIMATE CT: $0 \le a_n \le b_n$, $R_n = s - s_n = a_{n+1}+a_{n+2}+\dots$ Then $0 \le R_n \le T_n$

$s = \sum a_n$, $t = \sum b_n$, $s_n = a_1+\dots+a_n$, $t_n = b_1+\dots+b_n$, $T_n = t - t_n = b_{n+1}+b_{n+2}+\dots$

e.g. $a_n = \frac{1}{2^n+1}$, $b_n = \frac{1}{2^n}$: $R_n \le T_n = \frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}+\dots = \frac{1/2^{n+1}}{1-1/2} = \frac{1}{2^n} < 0.001$ for $n \ge 10$.

11.5 AST: If (i) $b_n \ge b_{n+1}$ (ii) $\lim_{n\to\infty} b_n = 0$ then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is convergent. e.g. $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is CONV bc. $1/\sqrt{n}$ decreasing, $\lim_{n\to\infty} 1/\sqrt{n} = 0$

ESTIMATION AST: $|R_n| = |s - s_n| = |b_{n+1} - b_{n+2} + b_{n+3} - \dots| \le b_{n+1}$ | e.g. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$, $|R_n| \le \frac{1}{n!} < 0.001$ for $n \ge 7$

11.6 $\sum a_n$ is absolutely convergent if $\sum|a_n|$ is convergent

— " — conditionally convergent if convergent but not abs. conv.

Thm If $\sum_n a_n$ is absolutely conv., then it is convergent.

e.g. $\sum_n\frac{(-1)^n}{2^n}$ is ~~abs.~~ conv. bc. $\sum\left|\frac{(-1)^n}{2^n}\right| = \sum\frac{1}{2^n}$ is a convergent geometric series.

$\sum_n\frac{\cos(n\pi)}{n} = \sum\frac{(-1)^n}{n}$ is conv. (by AST) but not ABS CONV.

ROOT/RATIO TEST: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. If (i) $L<1$, $\sum_n a_n$ is ABS. CONV (ii) $L>1$, $\sum_n a_n$ is DIVERGENT (iii) $L=1$, NO CONCLUSION! (NOTE: L should always be ≥ 0)

USE ROOT for powers, RATIO for products/factorials. ROOT: $\lim_{n\to\infty}\sqrt[n]{n} = 1$, $\lim_{n\to\infty}\sqrt[n]{\text{polynomial}} = 1$

$\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$: use ROOT: $\sqrt[n]{\frac{n}{(\ln n)^n}} = \frac{\sqrt[n]{n}}{\ln n} \to \frac{1}{\infty} = 0 \Rightarrow$ CONV

RATIO → $\sum_{n=1}^{\infty}\frac{n!}{e^{n^2}}$: $\frac{\frac{(n+1)!}{e^{(n+1)^2}}}{\frac{n!}{e^{n^2}}} = \frac{n+1}{e^{(n+1)^2-n^2}} = \frac{n+1}{e^{2n+1}} \to 0 \Rightarrow$ CONV.

$\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$: $\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \left(1+\frac{1}{n}\right)^n \to e > 1 \Rightarrow$ DIV

$\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)^n$: $\sqrt[n]{\left(\sqrt[n]{2}-1\right)^n} = \sqrt[n]{2}-1 \to 1-1 = 0 \Rightarrow$ CONV

$\sum_{n=0}^{\infty}\frac{n!}{2\cdot5\cdot8\cdots(3n+2)}$: $\frac{(n+1)!}{2\cdot5\cdot8\cdots(3n+2)(3n+5)}\cdot\frac{2\cdot5\cdot8\cdots(3n+2)}{n!} = \frac{n+1}{3n+5} \to \frac{1}{3} \Rightarrow$ CONV.