# Midterm #2 Review

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## 11 Series

**Theorem 11.1.** *(Geometric series) The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

*is convergent if* $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

*If* $|r| \geq 1$*, the geometric series is divergent.*

**Example 11.2.** *Is the series* $\sum_{n=1}^{\infty} 2^{2n+1}5^{2-n}$ *convergent or divergent?*

*Solution.* We have

$$\sum_{n=1}^{\infty} 2^{2n+1}5^{2-n} = \sum_{n=1}^{\infty} 40 \left(\frac{4}{5}\right)^{n-1}$$

We take $a = 40$, $r = 4/5$ in the previous theorem, and conclude that the series is convergent. The sum of the series is then

$$40 \cdot \frac{1}{1 - \frac{4}{5}} = 200.$$

$\square$

**Example 11.3.** *Express the number* $1.2\overline{31} = 1.2313131\cdots$ *as a ratio of integers.*

*Solution.* We have

$$\begin{aligned} 1.2\overline{31} &= 1.2 + \frac{31}{1000} + \frac{31}{10^5} + \frac{31}{10^7} + \cdots \\ &= \frac{6}{5} + \sum_{n=1}^{\infty} \frac{31}{10^{2n+1}} \\ &= \frac{6}{5} + \frac{31}{1000} \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^{n-1} \\ &= \frac{6}{5} + \frac{31}{1000} \cdot \frac{1}{1 - \frac{1}{100}} \\ &= \frac{6}{5} + \frac{31}{990} = \frac{1219}{990}. \end{aligned}$$

$\square$

**Example 11.4.** *(Telescoping sums) Determine the sum of the following series.*

*1.* $\displaystyle\sum_{n=3}^{\infty} \frac{4}{n^2-4}$.

*2.* $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

*Solution.* For the 1st example, we have

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{4}{n^2-4} &= \sum_{n=3}^{\infty} \left( \frac{1}{n-2} - \frac{1}{n+2} \right) \\
&= \left(\frac{1}{1} - \frac{1}{5}\right) + \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.
\end{aligned}$$

(we got the last equality by noticing that all the terms starting with $1/5$ appear once with a $+$ and once with a $-$ sign).

For the 2nd example, we have

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{(n+1)-1}{(n+1)!} = \sum_{n=1}^{\infty} \left( \frac{1}{n!} - \frac{1}{(n+1)!} \right) = \left( \frac{1}{1!} - \frac{1}{2!} \right) + \left( \frac{1}{2!} - \frac{1}{3!} \right) + \cdots = 1.$$

□

**Theorem 11.5.** *(Test for divergence) If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum a_n$ *is divergent.*

**Example 11.6.** *Is the series* $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^4}$ *convergent or divergent? What about* $\displaystyle\sum_{n=1}^{\infty} \frac{(-n!)^n}{n^{4n}}$*?*

*Solution.* Using l'Hôpital's rule we have

$$\lim_{n\to\infty} \frac{e^n}{n^4} = \lim_{n\to\infty} \frac{e^n}{4n^3} = \lim_{n\to\infty} \frac{e^n}{12n^2} = \lim_{n\to\infty} \frac{e^n}{24n} = \lim_{n\to\infty} \frac{e^n}{24} = \infty,$$

hence by the Test of divergence we conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^4}$ is divergent.

We have

$$\lim_{n\to\infty} \frac{n!}{n^4} = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \frac{n-3}{n} \cdot (n-4)! = \lim_{n\to\infty} (n-4)! = \infty.$$

It follows that

$$\lim_{n\to\infty} \frac{(-n!)^n}{n^{4n}} = \lim_{n\to\infty} (-1)^n \left( \frac{n!}{n^4} \right)^n, \text{ hence does not exist.}$$

From the test for Divergence, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{(-n!)^n}{n^{4n}}$ is divergent. □

**Example 11.7.** *Determine whether the series is absolutely convergent, conditionally convergent, or divergent*

1. $\sum_{n=1}^{\infty}(-1)^n \dfrac{n}{\sqrt{n^3+1}}$.

2. $\sum_{n=1}^{\infty} \arcsin(1/n^2)$

*Solution.* (1) Let

$$a_n = \frac{n}{\sqrt{n^3+1}} = \frac{1}{\sqrt{\frac{n^3+1}{n^2}}} = \frac{1}{\sqrt{n+\frac{1}{n^2}}}.$$

The function $f(x) = x + \dfrac{1}{x^2}$ is eventually increasing:

$$f'(x) = 1 - \frac{2}{x^3} \geq 0 \text{ for } x \geq \sqrt[3]{2}.$$

Therefore $\frac{1}{\sqrt{f(x)}}$ is eventually decreasing, and the same is true for the sequence $a_n$. It is clear that $\lim_{n\to\infty} a_n = 0$, so the Alternating Series Test yields the convergence of the series $\sum_{n=1}^{\infty}(-1)^n a_n$.

To test the absolute convergence, we use the Limit Comparison Theorem. We compare the series $\sum a_n$ with $\sum \dfrac{1}{\sqrt{n}}$. We have

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n\sqrt{n}}{\sqrt{n^3+1}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+1}} = 1.$$

Since $\sum \dfrac{1}{\sqrt{n}}$ is divergent by the $p$-series Test, we get the divergence of $\sum a_n$. This says that the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is not absolutely convergent, and since we've seen it is convergent, it has to be conditionally convergent.

(2) This series has positive terms, so convergence and absolute convergence are equivalent. We test its convergence using the Limit Comparison Theorem. We compare $\sum \arcsin(1/n^2)$ with $\sum \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\arcsin(1/n^2)}{1/n^2} \overset{x=1/n^2}{=} \lim_{x\to 0} \frac{\arcsin\ x}{x} \overset{\text{l'Hôpital}}{=} \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} = 1.$$

Since $\sum \frac{1}{n^2}$ is convergent by the $p$-series test, we conclude that $\sum \arcsin(1/n^2)$ is also convergent. □

**Example 11.8.** *Find the values of $c$ for which the following series converges.*

$$\sum_{n=1}^{\infty}\left(\frac{c}{n} - \frac{1}{n+1}\right)$$

*Solution.* We have

$$\sum_{n=1}^{\infty}\left(\frac{c}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty}\left(\frac{c-1}{n} + \frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty}\left(\frac{c-1}{n} + \frac{1}{n(n+1)}\right)$$

The $p$-series test (or telescoping) yields the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Since the harmonic series is divergent, we see that the series

$$\sum_{n=1}^{\infty} \left( \frac{c}{n} - \frac{1}{n+1} \right)$$

is convergent for $c = 1$, and divergent otherwise. $\square$

**Example 11.9.** *Find a power series representation for the function and determine the interval (for the 1st example) and radius (for both) of convergence*

1. $f(x) = \dfrac{x^3 + x^2 - x + 1}{x^4 - 1}$.
2. $f(x) = x^2 \arcsin(x)$.

*Solution.* (1) We have

$$f(x) = (-1+x-x^2-x^3)\frac{1}{1-x^4} = (-1+x-x^2-x^3)\sum_{n=0}^{\infty} x^{4n} = \sum_{n=0}^{\infty}(-x^{4n}+x^{4n+1}-x^{4n+2}-x^{4n+3}).$$

Since the radius of convergence for $\sum x^{4n}$ is 1, and the interval of convergence is $(-1, 1)$, the same is true for the power series representation for $f(x)$.

(2) It suffices to determine the power series representation for $\arcsin(x)$. We have

$$\arcsin(x)' = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n.$$

Integrating this equality we get

$$\arcsin(x) = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \frac{x^{2n+1}}{2n+1} + c.$$

Plugging in $x = 0$ we get $c = 0$. The radius of convergence for the binomial series is 1, hence the same is true for the power series of $\arcsin(x)$. We get

$$f(x) = x^2 \arcsin(x) = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \frac{x^{2n+3}}{2n+1}.$$

$\square$

**Example 11.10.** *Determine the Maclaurin series for $f(x) = \sinh\ x$ and prove that it represents $f(x)$ for all values of $x$.*

*Solution.* We have

$$f(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left( \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

The formula for $f^{(n)}(x)$ depends only on the parity of $n$, hence we can find a uniform bound for all the derivatives of $\sinh\ x$ on any interval. Taylor's inequality then proves that

$$f(x) = \lim_{n\to\infty} T_n(x) \text{ for all } x$$

i.e. the Maclaurin series for $f(x)$ represents $f(x)$ for all values of $x$. $\square$

**Example 11.11.** *Find the first three nonzero terms in the Maclaurin series for*

$$f(x) = \frac{e^x}{1+x}.$$

*Solution.* We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

and

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + \cdots$$

Therefore

$$\begin{aligned}\frac{e^x}{1+x} &= (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots)(1 - x + x^2 - x^3 + \cdots) \\ &= 1 \cdot 1 + x(1 \cdot (-1) + 1 \cdot 1) + x^2(1 \cdot 1 + 1 \cdot (-1) + (1/2) \cdot 1) \\ &+ x^3(1 \cdot (-1) + 1 \cdot 1 + (1/2) \cdot (-1) + (1/6) \cdot 1) + \cdots \\ &= 1 + \frac{x^2}{2} - \frac{x^3}{3} + \cdots\end{aligned}$$

□

**Example 11.12.** *For the given function $f$, do the following.*

*(a) Approximate $f$ by a Taylor polynomial with degree $n$ at the number $a$.*

*(b) Use Taylor's inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when $x$ lies in the given interval.*

$$f(x) = \ln(1 + 2x),\ a = 1,\ n = 3,\ 0.5 \le x \le 1.5$$

*Solution.* We have

$$\begin{gathered}f(1) = \ln(3), \\ f'(x) = \frac{2}{1+2x}, \quad f'(1) = \frac{2}{3}, \\ f''(x) = \frac{-4}{(1+2x)^2}, \quad f''(1) = \frac{-4}{9}, \\ f'''(x) = \frac{16}{(1+2x)^3}, \quad f'''(1) = \frac{16}{27}, \\ f^{(4)}(x) = \frac{-96}{(1+2x)^4}.\end{gathered}$$

It follows that

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(x)}{n!}(x-1)^n = \ln(3) + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3.$$

We estimate the accuracy of the approximation $f(x) \approx T_3(x)$ via Taylor's inequality:

$$|f(x) - T_3(x)| \le \frac{M}{4!}|x-1|^4,$$

where $M$ is such that

$$|f^{(4)}(x)| \leq M \text{ for } 0.5 \leq x \leq 1.5$$

We have

$$|f^{(4)}(x)| = \frac{96}{(1+2x)^4} \leq \frac{96}{(1+2\cdot 0.5)^4} = \frac{96}{2^4} = 6,$$

so we can take $M = 6$. Also $|x-1| \leq 0.5 = 1/2$, hence $|x-1|^4 \leq 1/16$. We get

$$|f(x) - T_3(x)| \leq \frac{6}{4!} \cdot \frac{1}{16} = \frac{1}{64}, \text{ for } 0.5 \leq x \leq 1.5$$

$\square$