Worksheet 10

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1. Let c be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c}$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than the exponent 1 for natural growth.

(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.

(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t\to T^-} y(t) = \infty$.

(c) An especially prolific breed of rabbits has the growth term $ky^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

2. To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)dx}$ and integrate both sides. Solve the following linear differential equations and initial value problems

(a) $y' = x + 5y$.

(b) $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

(c) $x^2y' + xy = 1$, $x > 0$, $y(1) = 2$.

(d) $t^3 - t\dfrac{dy}{dt} = 2y$, $t > 0$, $y(1) = 0$.

(e) $t\ln(t)\dfrac{dr}{dt} + r = te^t$.

(f) $(x^2+1)\dfrac{dy}{dx} + 3x(y-1) = 0,\ y(0) = 2$.

3. A tank contains 100L of water. A solution with a salt concentration of 0.4kg/L is added at a rate of 5L/min. The solution is kept mixed and is drained from the tank at a rate of 3L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, show that y satisfies the differential equation

$$\frac{dy}{dt} = 2 - \frac{3y}{100+2t}.$$

Solve this equation and find the concentration after 20 minutes.

4. A Bernoulli differential equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

For $n \neq 1$, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Use this method to solve

$$xy' + y = -xy^2.$$

Solve the differential equation

5. $y'' - 4y' + y = 0.$

6. $y'' + 3y' = 0.$

Solve the initial-value problem

7. $2y'' + 5y' - 3y = 0$, $y(0) = 1$, $y'(0) = 4$.

8. $y'' - 2y' + 5y = 0$, $y(\pi) = 0$, $y'(\pi) = 2$.

Solve the boundary-value problem, if possible.

9. $y'' + 4y' + 13y = 0$, $y(0) = 2$, $y(\pi/2) = 1$.

10. $y'' - 6y' + 9y = 0$, $y(0) = 1$, $y(1) = 0$.

11. If a, b, c are all positive constants and $y(x)$ is a solution of the differential equation $ay'' + by' + cy = 0$, show that $\lim_{x\to\infty} y(x) = 0$.