Worksheet 10 - Solutions

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April 19, 2010

1. (a) By separating the variables and integrating we get

$$\int \frac{dy}{y^{1+c}} = \int k dt$$

which is equivalent to

$$\frac{y^{-c}}{-c} = kt + A$$

for some constant A. The initial condition $y(0) = y_0$ yields

$$\frac{y_0^{-c}}{-c} = A$$

so

$$\frac{y^{-c}}{-c} = kt + \frac{y_0^{-c}}{-c}.$$

Multiplying by $-c$ we obtain

$$y^{-c} = -ckt + y_0^{-c},$$

or equivalently

$$y = (-ckt + y_0^{-c})^{-1/c} = \frac{1}{(-ckt + y_0^{-c})^{1/c}}.$$

(b) Let $T = y_0^{-c}/ck$. Clearly

$$\lim_{t \to T^-} -ckt + y_0^{-c} = -ckT + y_0^{-c} = 0$$

and $-ckt + y_0^{-c} > 0$ for $t < T$, thus

$$\lim_{t \to T^-} = \infty.$$

(c) In this case we have $c = 0.01$ and an initial population of $y_0 = 2$ rabbits. We also know that $y(3) = 16$, which by using the equality

$$y^{-c} = -ckt + y_0^{-c}$$

shows that

$$16^{-0.01} = -3ck + 2^{-0.01},$$

i.e.

$$ck = \frac{2^{-0.01} - 16^{0.01}}{3}$$

Doomsday is given by

$$T = \frac{y_0^{-c}}{ck} = \frac{2^{-0.01}}{2^{-0.01} - 16^{-0.01}} \cdot 3 = \frac{1}{1 - 8^{-0.01}} \cdot 3 \approx 146$$

which is approximately 12 years and 2 months.

2. Recall that to solve a linear equation we multiply both sides by the integrating factor and get

$$(I(x)y)' = I(x)Q(x).$$

Integrating this equality yields

$$I(x)y = \int I(x)Q(x)dx,$$

or equivalently

$$y = \frac{\int I(x)Q(x)dx}{I(x)}.$$

(a) We rewrite the equation as

$$y' - 5y = x$$

so $P(x) = -5$, $Q(x) = x$,

$$I(x) = e^{\int -5dx} = e^{-5x}.$$

We have

$$ye^{-5x} = \int e^{-5x}xdx = e^{-5x}\left(-\frac{x}{5} - \frac{1}{25}\right) + C$$

(you need to use integration by parts to get the last equality). Dividing by $I(x) = e^{-5x}$ we obtain

$$y = -\frac{x}{5} - \frac{1}{25} + Ce^{5x}.$$

(b) We rewrite the equation as

$$y' + 3x^2y = 6x^2$$

so $P(x) = 3x^2$, $Q(x) = 6x^2$,

$$I(x) = e^{\int 3x^2dx} = e^{x^3}.$$

We have

$$ye^{x^3} = \int e^{x^3}6x^2dx = 2e^{x^3} + C$$

(you need to make the substitution $u = x^3$ to obtain the last equality). Dividing by $I(x) = e^{x^3}$ we obtain

$$y = 2 + Ce^{-x^3}.$$

(c) We divide both sides by x^2 and rewrite the equation as

$$y' + \frac{1}{x}y = \frac{1}{x^2}$$

so $P(x) = 1/x$, $Q(x) = 1/x^2$,

$$I(x) = e^{\int 1/xdx} = e^{\ln(x)} = x$$

(we don't need absolute values because $x > 0$). We have

$$yx = \int x \frac{1}{x^2} dx = \ln(x) + C$$

Dividing by $I(x) = x$ we obtain

$$y = \frac{\ln(x) + C}{x}.$$

The initial condition $y(1) = 2$ yields

$$2 = \frac{0 + C}{1}, \text{ i.e. } C = 2,$$

thus

$$y = \frac{\ln(x) + 2}{x}.$$

(d) We divide both sides by t and rewrite the equation as

$$y' + \frac{2}{t} y = t^2$$

so $P(x) = 2/t$, $Q(t) = t^2$,

$$I(t) = e^{\int 2/t dt} = e^{2\ln(t)} = t^2$$

(we don't need absolute values because $t > 0$). We have

$$yt^2 = \int t^2 \cdot t^2 dt = \frac{t^5}{5} + C$$

Dividing by $I(t) = t^2$ we obtain

$$y = \frac{t^3}{5} + \frac{C}{t^2}.$$

The initial condition $y(1) = 0$ yields

$$0 = \frac{1}{5} + C, \text{ i.e. } C = -1/5,$$

thus

$$y = \frac{t^3}{5} - \frac{1}{5t^2}.$$

(e) We must have $t > 0$ in order for the $\ln(t)$ term to make sense. We divide both sides by $t\ln(t)$ and rewrite the equation as

$$r' + \frac{1}{t\ln(t)} r = \frac{e^t}{\ln(t)}$$

so $P(x) = 1/(t\ln(t))$, $Q(t) = e^t/\ln(t)$,

$$I(t) = e^{\int 1/(t\ln(t))dt} = e^{\ln(\ln(t))} = \ln(t)$$

(we used the substitution $u = \ln(t)$ to integrate $1/(t\ln(t))$; also, we don't need absolute values because $t > 0$). We have

$$y\ln(t) = \int \ln(t) \cdot \frac{e^t}{\ln(t)} dt = \int e^t dt = e^t + C$$

Dividing by $I(t) = \ln(t)$ we obtain

$$y = \frac{e^t + C}{\ln(t)}.$$

(f) See the 2nd and 3rd solutions to the first problem on Quiz #10. The solution of the differential equation is

$$y = 1 + \frac{C}{(x^2+1)^{3/2}}$$

and the initial condition $y(0) = 2$ yields

$$2 = 1 + C, \text{ i.e. } C = 1,$$

so that

$$y = 1 + \frac{1}{(x^2+1)^{3/2}}.$$

3. If we denote by $A(t)$ the total mass of the mixture at time t, we see that $A(t) = 100 + 2t$, since A increases at a rate of $5 - 3 = 2$L/min. If we let $c(t)$ denote the concentration of salt at time t, then

$$c(t) = \frac{y(t)}{A(t)} = \frac{y}{100+2t}.$$

The change in the amount of salt in the mixture is measured, on the one hand by the derivative of y, and on the other hand by the difference between the amount of salt that goes into the mixture, which is $5 \cdot 0.4 = 2$, and the amount of salt that goes out of the mixture, which is $3c(t)$. It follows that

$$y' = 2 - 3c(t) = 2 - \frac{3y}{100+2t}.$$

Rewriting the equation as

$$y' + \frac{3y}{100+2t} = 2,$$

we see that it is a linear equation, with $P(t) = 3/(100+2t)$ and $Q(t) = 2$. We get

$$I(t) = e^{\int 3/(100+2t)dt} = e^{3/2 \cdot \ln(100+2t)} = (100+2t)^{3/2}.$$

It follows that

$$yI(t) = \int I(t) \cdot Q(t)dt = \frac{2}{5}(100+2t)^{5/2} + C.$$

Dividing by $I(t) = (100+2t)^{3/2}$ we get

$$y = \frac{2(100+2t)}{5} + \frac{C}{(100+2t)^{3/2}}.$$

Since at time 0 the tank contains only water, this means that $y(0) = 0$, so

$$0 = 40 + \frac{C}{1000}, \text{ i.e. } C = -40000.$$

We get

$$y = \frac{2(100+2t)}{5} - \frac{40000}{(100+2t)^{3/2}}.$$

The concentration of salt after 20 minutes is then

$$c(20) = \frac{y(20)}{A(20)} = \frac{31.85}{140} \approx 0.2275\text{kg/L}.$$

4. Notice that for $n = 0$ the equation is linear, and for $n = 1$ it is equivalent to

$$y' = (Q(x) - P(x))y$$

which is separable, so we know how to solve it in both cases.

Assuming now that $n \neq 1$, and dividing both sides of the Bernoulli equation by y^n, we obtain

$$y'y^{-n} + P(x)y^{1-n} = Q(x).$$

If we make the substitution $u = y^{1-n}$, then $u' = (1-n)y^{-n}y'$, or equivalently

$$y'y^{-n} = \frac{u'}{1-n}.$$

We then get a differential equation in u:

$$\frac{u'}{1-n} + P(x)u = Q(x)$$

which after multiplication by $(1-n)$ takes the form of a linear equation

$$u' + (1-n)P(x)u = (1-n)Q(x).$$

To solve $xy' + y = -xy^2$, we first divide by x to get

$$y' + \frac{1}{x}y = -y^2$$

so $P(x) = 1/x$ and $Q(x) = -1$ (the notation is as above). This is a Bernoulli equation with $n = 2$, so the substitution $u = y^{1-2} = y^{-1}$ yields the linear equation

$$u' - \frac{1}{x}u = (-1) \cdot (-1).$$

For this one we have

$$I(x) = e^{\int -1/x dx} = e^{-\ln(x)} = \frac{1}{x}$$

so

$$u \cdot \frac{1}{x} = \int \frac{1}{x} \cdot 1 dx = \ln(x) + C.$$

This shows that

$$u = x(\ln(x) + C)$$

and therefore

$$y = u^{-1} = \frac{1}{x(\ln(x) + C)}.$$

5. The auxiliary equation $r^2 - 4r + 1$ has roots $2 \pm \sqrt{3}$ which are distinct real numbers, so the solution of the differential equation is given by

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}.$$

6. The auxiliary equation $r^2 + 3r$ has roots 0 and -3 which are distinct real numbers, so the solution of the differential equation is given by

$$y = c_1 e^{0 \cdot x} + c_2 e^{-3x} = c_1 + c_2 e^{-3x}.$$

7. The auxiliary equation $2r^2+5r-3$ has roots $1/2$ and -3 which are distinct real numbers, so the solution of the differential equation is given by

$$y = c_1 e^{x/2} + c_2 e^{-3x}.$$

The initial conditions yield

$$1 = y(0) = c_1 + c_2 \text{ and } 4 = y'(0) = c_1/2 - 3c_2.$$

Solving the system of equations we get $c_1 = 2$ and $c_2 = -1$, so

$$y = 2e^{x/2} - e^{-3x}.$$

8. The auxiliary equation r^2-2r+5 has roots $1 \pm 2i$ which are complex conjugate numbers, so the solution of the differential equation is given by

$$y = e^x(c_1 \cos(2x) + c_2 \sin(2x)).$$

The initial conditions yield

$$0 = y(\pi) = e^{\pi} c_1 \text{ and } 2 = y'(\pi) = e^{\pi}(c_1 + 2c_2).$$

Solving the system of equations we get $c_1 = 0$ and $c_2 = e^{-\pi}$, so

$$y = e^{x-\pi} \sin(2x).$$

9. The auxiliary equation $r^2 + 4r + 13$ has roots $-2 \pm 3i$ which are complex conjugate numbers, so the solution of the differential equation is given by

$$y = e^{-2x}(c_1 \cos(3x) + c_2 \sin(3x)).$$

The initial conditions yield

$$2 = y(0) = c_1 \text{ and } 1 = y(\pi/2) = e^{-\pi}(-c_2).$$

Solving the system of equations we get $c_1 = 2$ and $c_2 = -e^{\pi}$, so

$$y = e^{-2x}(2\cos(3x) - e^{\pi} \sin(3x)).$$

10. The auxiliary equation r^2-6r+9 has double root $r = 3$, so the solution of the differential equation is given by

$$y = e^{3x}(c_1 + c_2 x).$$

The initial conditions yield

$$1 = y(0) = c_1 \text{ and } 0 = y(1) = e^3(c_1 + c_2).$$

Solving the system of equations we get $c_1 = 1$ and $c_2 = -1$, so

$$y = e^{3x}(1 - x).$$

11. The auxiliary equation $ar^2 + br + c = 0$ has roots given by the quadratic formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We have three cases:

<u>*Case 1:*</u> r_1, r_2 are real and distinct. In this case they have to be strictly negative numbers (if they were nonnegative, we would get $0 = ar_i^2 + br_i + c \geq c > 0$ which is imposible). The solution of the differential equation has the form

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x},$$

but since $r_1, r_2 < 0$, $\lim_{x\to\infty} r_i x = -\infty$ and therefore

$$\lim_{x\to\infty} e^{r_1 x} = \lim_{x\to\infty} e^{r_2 x} = e^{-\infty} = 0$$

thus

$$\lim_{x\to\infty} y(x) = 0 + 0 = 0.$$

<u>*Case 2:*</u> $r_1 = r_2 = r$ is a double root. In this case r has to be strictly negative for the same reason as in the preceding case. The solution of the differential equation has the form

$$y = e^{rx}(c_1 + c_2 x),$$

but since $r < 0$, $\lim_{x\to\infty} rx = -\infty$ and therefore

$$\lim_{x\to\infty} e^{rx} = 0$$

and

$$\lim_{x\to\infty} xe^{rx} = \lim_{x\to\infty} \frac{x}{e^{(-r)x}} \overset{l'H}{=} \lim_{x\to\infty} \frac{1}{-re^{(-r)x}} = \frac{1}{-\infty} = 0.$$

It follows that

$$\lim_{x\to\infty} y(x) = 0 + 0 = 0.$$

<u>*Case 3:*</u> r_1, r_2 are complex conjugate $(\alpha \pm \beta)$. Analyzing the quadratic formula, we see that we must have

$$\alpha = \frac{-b}{2a}$$

which is a negative number, since $a, b > 0$. The solution of the differential equation has the form

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)),$$

but since $\alpha < 0$, $\lim_{x\to\infty} \alpha x = -\infty$ and therefore

$$\lim_{x\to\infty} e^{\alpha x} = 0.$$

The function $c_1 \cos(\beta x) + c_2 \sin(\beta x)$ is bounded since sin and cos are bounded, so we can conclude that

$$\lim_{x\to\infty} y(x) = 0.$$

Since the above three cases exhaust all possibilities for r_1, r_2, we are done!