Worksheet 11 - Solutions

Claudiu Raicu

April 26, 2010

1. We first solve the complementary equation $y'' + y = 0$. This has auxiliary equation $r^2 + 1$ with roots $\pm i$, so its solutions are given by

$$y_c = c_1 \cos(x) + c_2 \sin(x).$$

We split the problem of finding a particular solution to our original equation into two parts, corresponding to the two terms e^x and x^3. We look for y_{p_1}, y_{p_2} solutions of

$$y'' + y = e^x \text{ and } y'' + y = x^3$$

respectively. Taking

$$y_{p_1} = Ae^x$$

we get

$$Ae^x + Ae^x = e^x,$$

so $A = 1/2$ and $y_{p_1} = e^x/2$. Taking

$$y_{p_2} = Bx^3 + Cx^2 + Dx + E$$

we get

$$(6Bx + 2C) + (Bx^3 + Cx^2 + Dx + E) = x^3.$$

Equating the coefficients of the various powers of x we obtain

$$2C + E = 0, \quad 6B + D = 0, \quad C = 0, \quad B = 1$$

which gives $D = -6$ and $E = 0$, so

$$y_{p_2} = x^3 - 6x.$$

It follows that a particular solution to our equation is given by

$$y_p = y_{p_1} + y_{p_2} = \frac{e^x}{2} + x^3 - 6x$$

and the general solution is

$$y = y_p + y_c = \frac{e^x}{2} + x^3 - 6x + c_1 \cos(x) + c_2 \sin(x).$$

2. We first solve the complementary equation $y'' - 4y = 0$. This has auxiliary equation $r^2 - 4$ with roots ± 2, so its solutions are given by

$$y_c = c_1 e^{2x} + c_2 e^{-2x}.$$

We now look for a particular solution y_p of our original equation of the form

$$y_p = e^x (A\cos(x) + B\sin(x)).$$

We have

$$\begin{aligned} y_p' &= e^x (A\cos(x) + B\sin(x) - A\sin(x) + B\cos(x)) \\ &= e^x [(A+B)\cos(x) + (B-A)\sin(x)]. \end{aligned}$$

$$\begin{aligned} y_p'' &= e^x [(A+B)\cos(x) + (B-A)\sin(x) - (A+B)\sin(x) + (B-A)\cos(x)] \\ &= e^x (2B\cos(x) - 2A\sin(x)). \end{aligned}$$

We must have

$$\begin{aligned} e^x \cos(x) = y_p'' - 4y_p &= e^x [(2B\cos(x) - 2A\sin(x)) - 4(A\cos(x) + B\sin(x))] \\ &= e^x [(2B - 4A)\cos(x) + (-2A - 4B)\sin(x)] \end{aligned}$$

which yields

$$2B - 4A = 1, \quad -2A - 4B = 0.$$

Solving the system of equations we get $A = -1/5$ and $B = 1/10$, so

$$y_p = e^x \cdot \frac{-2\cos(x) + \sin(x)}{10}.$$

It follows that the general solution is

$$y = y_p + y_c = e^x \cdot \frac{-2\cos(x) + \sin(x)}{10} + c_1 e^{2x} + c_2 e^{-2x}.$$

To determine c_1, c_2 we use the initial conditions: $y(0) = 1$ yields

$$c_1 + c_2 = \frac{6}{5}$$

and $y'(0) = 2$ yields

$$2c_1 - 2c_2 = \frac{21}{10}.$$

Solving the system of equations we get $c_1 = 9/8$ and $c_2 = 3/40$, so

$$y = e^x \cdot \frac{-2\cos(x) + \sin(x)}{10} + \frac{9}{8} e^{2x} + \frac{3}{40} e^{-2x}.$$

3. We first solve the complementary equation $y'' + y = 0$. This has auxiliary equation $r^2 + 1$ with roots $\pm i$, so its solutions are given by

$$y_c = c_1 \cos(x) + c_2 \sin(x).$$

We now vary the parameters to get a particular solution y_p of our original equation. We take

$$y_p = u_1 y_1 + u_2 y_2$$

where $y_1 = \cos(x)$, $y_2 = \sin(x)$ and u_1', u_2' satisfy the system of equations

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = \sec^3(x) \end{cases}$$

We rewrite the system as

$$\begin{cases} u_1'\cos(x) + u_2'\sin(x) = 0 \\ u_1'(-\sin(x)) + u_2'\cos(x) = \sec^3(x) \end{cases}$$

Multiplying the first equation by $\sin(x)$ and the second one by $\cos(x)$, and adding, we obtain

$$u_2' = u_2'(\sin^2(x) + \cos^2(x)) = \sec^2(x),$$

thus

$$u_2 = \int \sec^2(x)dx = \tan(x).$$

We have

$$u_1' = -u_2'\tan(x) = -\tan(x)\sec^2(x)$$

so

$$u_1 = \int(-\tan(x)\sec^2(x))dx = -\frac{\tan^2(x)}{2}$$

(where the last equality follows by making the substitution $u = \tan(x)$). Substituting back into the formula of y_p we get

$$y_p = -\frac{\tan^2(x)}{2}\cos(x) + \tan(x)\cdot\sin(x) = \frac{\sin^2(x)}{2\cos(x)}.$$

The general solution is therefore

$$y = y_p + y_c = \frac{\sin^2(x)}{2\cos(x)} + c_1\cos(x) + c_2\sin(x).$$

4. We first solve the complementary equation $y'' + 3y' + 2y = 0$. This has auxiliary equation $r^2 + 3r + 2$ with roots -1 and -2, so its solutions are given by

$$y_c = c_1e^{-x} + c_2e^{-2x}.$$

We now vary the parameters to get a particular solution y_p of our original equation. We take

$$y_p = u_1y_1 + u_2y_2$$

where $y_1 = e^{-x}$, $y_2 = e^{-2x}$ and u_1', u_2' satisfy the system of equations

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = \sin(e^x) \end{cases}$$

We rewrite the system as

$$\begin{cases} u_1'e^{-x} + u_2'e^{-2x} = 0 \\ u_1'(-e^{-x}) + u_2'(-2e^{-2x}) = \sin(e^x) \end{cases}$$

Adding the two equations we obtain

$$-u_2' e^{-2x} = \sin(e^x),$$

or equivalently

$$u_2' = -e^{2x}\sin(e^x).$$

It follows that

$$u_2 = \int (-e^{2x}\sin(e^x))dx = e^x\cos(e^x) - \sin(e^x)$$

(this follows from the substitution $u = e^x$). We have

$$u_1' = -u_2' e^{-x} = e^x \sin(e^x)$$

so

$$u_1 = \int (e^x \sin(e^x))dx = -\cos(e^x)$$

(again by the substitution $u = e^x$). Substituting back into the formula of y_p we get

$$y_p = -\cos(e^x)e^{-x} + (e^x\cos(e^x) - \sin(e^x))e^{-2x} = -\sin(e^x)e^{-2x}$$

The general solution is therefore

$$y = y_p + y_c = -\sin(e^x)e^{-2x} + c_1 e^{-x} + c_2 e^{-2x}.$$

7. The spring constant k is given by

$$k \cdot 0.25 = 13 \quad \Leftrightarrow \quad k = 52,$$

so the equation for displacement is

$$2x'' + 8x' + 52 = 0$$

(notice that $F(t) = 0$ since there's no force acting). Since the mass starts at equilibrium, we must have $x(0) = 0$. The velocity is the derivative of displacement, so $x'(0) = 0.5$.

We divide by 2 and rewrite the equation as

$$x'' + 4x' + 26 = 0.$$

The auxiliary equation $r^2 + 4r + 26$ has roots $-2 \pm i\sqrt{22}$, so its general solution is given by

$$x = e^{-2t}(c_1\cos(\sqrt{22}t) + c_2\sin(\sqrt{22}t)).$$

The initial conditions yield

$$0 = x(0) = c_1 \text{ and } 0.5 = x'(0) = -2c_1 + \sqrt{22}c_2,$$

so $c_1 = 0$, $c_2 = 1/(2\sqrt{22})$. We get

$$x(t) = \frac{1}{2\sqrt{22}}\sin(\sqrt{22}t)e^{-2t}.$$

8. Since the damping force is negligible, we may assume that $c = 0$. The equation for displacement is then

$$mx'' + kx = F(t)$$

which after dividing by m becomes

$$x'' + \omega^2 x = \frac{F_0}{m}\cos(\omega_0 t).$$

We first solve the complementary equation, whose auxiliary equation $r^2 + \omega^2 = 0$ has roots $\pm i\omega$. Its solutions are then given by

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

We now look for a particular solution x_p of the original equation. We take x_p of the form

$$x_p = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

(notice that if ω_0 was equal to ω this wouldn't work, and we would have had to replace x_p by tx_p). We want x_p to satisfy

$$x_p'' + \omega^2 x_p = \frac{F_0}{m}\cos(\omega_0 t).$$

Notice that since x_p is a linear combination of $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$, it satisfies the differential equation $x_p'' = -\omega_0^2 x_p$, so we can rewrite the above relation as

$$(\omega^2 - \omega_0^2)x_p = (\omega^2 - \omega_0^2)(A\cos(\omega_0 t) + B\sin(\omega_0 t)) = \frac{F_0}{m}\cos(\omega_0 t).$$

This says that $(\omega^2 - \omega_0^2)A = F_0/m$ and $(\omega^2 - \omega_0^2)B = 0$, yielding $B = 0$ and $A = F_0/(m \cdot (\omega^2 - \omega_0^2))$. We get

$$x_p = \frac{F_0}{m(\omega^2 - \omega_0^2)}\cos(\omega_0 t).$$

The general solution to the original equation is therefore

$$x = x_p + x_c = \frac{F_0}{m(\omega^2 - \omega_0^2)}\cos(\omega_0 t) + c_1\cos(\omega t) + c_2\sin(\omega t).$$

9. The differential equation for charge is

$$2Q'' + 24Q' + \frac{1}{0.005}Q = 12.$$

The initial conditions are $Q(0) = 0.001$ and $Q'(0) = I(0) = 0$. We divide by 2 and rewrite our equation as

$$Q'' + 12Q' + 100Q = 6.$$

We first solve the complementary equation, whose auxiliary equation $r^2 + 12r + 100 = 0$ has roots $-6 \pm 8i$. Its solutions are then given by

$$Q_c = e^{-6t}(c_1\cos(8t) + c_2\sin(8t)).$$

We now look for a particular solution Q_p of the nonhomogeneous equation. We take Q_p of the form

$$Q_p = A$$

which yields

$$100A = 6 \quad \Leftrightarrow \quad A = 3/50.$$

The general solution of the equation is then

$$Q = Q_p + Q_c = \frac{3}{50} + e^{-6t}(c_1 \cos(8t) + c_2 \sin(8t)).$$

The initial conditions yield

$$0.001 = Q(0) = \frac{3}{50} + c_1 \text{ and } 0 = Q'(0) = -6c_1 + 8c_2.$$

Solving the system of equations we get $c_1 = 1/1000 - 3/50 = -59/1000$ and $c_2 = (3/4)c_1 = -177/4000$, so

$$Q(t) = \frac{3}{50} - e^{-6t}\frac{236\cos(8t) + 177\sin(8t)}{4000}.$$

10. We look for a power series solution

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

This has

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

(a) We rewrite the equation as

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0,$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We reindex the first sum to make the exponent of x equal to n. That is, we replace n by $n+2$, so the initial value $n = 2$ becomes $n + 2 = 2 \Leftrightarrow n = 0$. We get

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^{n} + \sum_{n=1}^{\infty} n c_n x^{n} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Separating out the $n = 0$ terms in the first and third sums and combining the sums together, we get

$$c_2 + c_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1)c_{n+2} + n c_n + c_n\right) x^n = 0.$$

Equating the coefficients of the power series on the LHS to zero, we obtain

$$\begin{cases} 2c_2 + c_0 = 0 \\ (n+2)(n+1)c_{n+2} + (n+1)c_n = 0 \quad \text{ for } n \geq 1 \end{cases}$$

which is the same as

$$\begin{cases} c_2 = -c_0/2 \\ c_{n+2} = \frac{-1}{n+2}c_n = 0 \quad \text{ for } n \geq 1 \end{cases}$$

We see that there's no condition on c_0 and c_1, and that all the other terms can be calculated from the two as follows. We have $c_2 = -c_0/2$,

$$c_3 = \frac{-1}{3}c_1 = \frac{-1}{1 \cdot 3}c_1$$

$$c_4 = \frac{-1}{4}c_2 = \frac{(-1)^2}{2 \cdot 4}c_0$$

$$c_5 = \frac{-1}{5}c_3 = \frac{(-1)^2}{1 \cdot 3 \cdot 5}c_1$$

$$c_6 = \frac{-1}{6}c_2 = \frac{(-1)^3}{2 \cdot 4 \cdot 6}c_0$$

$$c_7 = \frac{-1}{7}c_2 = \frac{(-1)^3}{1 \cdot 3 \cdot 5 \cdot 7}c_1$$

$$\cdots$$

We see that there are two distinct patterns, one for the even terms, and one for the odd terms, namely

$$c_{2k} = \frac{(-1)^k}{2 \cdot 4 \cdots (2k)}c_0$$

and

$$c_{2k+1} = \frac{(-1)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}c_1,$$

and these formulas are accurate for all values of $k \geq 0$. We get

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} c_{2k}x^{2k} + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1} \\ &= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2 \cdot 4 \cdots (2k)}x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}x^{2k+1}. \end{aligned}$$

If you prefer, you can rewrite the coefficients c_{2k}, c_{2k+1} as

$$c_{2k} = \frac{(-1)^k}{2^k \cdot k!}c_0$$

and

$$c_{2k+1} = \frac{(-1)^k \cdot 2 \cdot 4 \cdot 6 \cdots (2k)}{1 \cdot 2 \cdot 3 \cdots (2k) \cdot (2k+1)}c_1 = \frac{(-2)^k \cdot k!}{(2k+1)!}c_1.$$

(b) We rewrite the equation as

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = x\sum_{n=0}^{\infty} c_n x^n = 0,$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} c_n x^{n+1}.$$

We reindex the sum on the LHS to make the exponent of x equal to $n+1$. That is, we replace n by $n+3$, so the initial value $n=2$ becomes $n+3=2 \Leftrightarrow n=-1$. We get

$$\sum_{n=-1}^{\infty} (n+3)(n+2)c_{n+3} x^{n+1} = \sum_{n=0}^{\infty} c_n x^{n+1}.$$

The $n=-1$ term on the LHS has to be 0, since there's no corresponding term on the RHS. Therefore

$$(-1+3)(-1+2)c_0 = 0 \Leftrightarrow c_0 = 0$$

and

$$\sum_{n=0}^{\infty}(n+3)(n+2)c_{n+3} x^{n+1} = \sum_{n=0}^{\infty} c_n x^{n+1}.$$

The coefficients of x^{n+1} have to match, therefore

$$(n+3)(n+2)c_{n+3} = c_n \text{ for } n \geq 0,$$

or equivalently

$$c_{n+3} = \frac{1}{(n+2)(n+3)} c_n \text{ for } n \geq 1.$$

We see that there is no condition on c_1, c_2, and all the other terms can be written in terms of the two. We have

$$\begin{aligned}
c_3 &= \frac{1}{2 \cdot 3} c_0 = 0 \\
c_4 &= \frac{1}{3 \cdot 4} c_1 \\
c_5 &= \frac{1}{4 \cdot 5} c_2 \\
c_6 &= \frac{1}{5 \cdot 6} c_3 = 0 \\
c_7 &= \frac{1}{6 \cdot 7} c_4 = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} c_1 \\
c_8 &= \frac{1}{7 \cdot 8} c_5 = \frac{1}{4 \cdot 5 \cdot 7 \cdot 8} c_2 \\
&\cdots
\end{aligned}$$

We see that there are three distinct patterns, corresponding to terms of the form $3k, 3k+1$ and $3k+2$ respectively, namely

$$c_{3k} = 0,$$

$$c_{3k+1} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3k)(3k+1)} c_1$$

and

$$c_{3k+2} = \frac{1}{4 \cdot 5 \cdot 7 \cdot 8 \cdots (3k+1)(3k+2)} c_2.$$

The formulas for c_{3k+1} and c_{3k+2} are accurate for $k \geq 1$ (for $k = 0$ they're ambiguous), so we will separate out the terms corresponding to $k = 0$ in the formulas below. We have

$$\begin{aligned}
y = \sum_{n=0}^{\infty} c_n x^n &= \sum_{k=0}^{\infty} c_{3k} x^{3k} + \sum_{k=0}^{\infty} c_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} c_{3k+2} x^{3k+2} \\
&= c_1 x + \sum_{k=1}^{\infty} c_{3k+1} x^{3k+1} + c_2 x^2 + \sum_{k=1}^{\infty} c_{3k+2} x^{3k+2} \\
&= c_1 x + c_2 x^2 + c_1 \sum_{k=1}^{\infty} \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3k)(3k+1)} x^{3k+1} \\
&\qquad + c_2 \sum_{k=1}^{\infty} \frac{1}{4 \cdot 5 \cdot 7 \cdot 8 \cdots (3k+1)(3k+2)} x^{3k+2}.
\end{aligned}$$

(c) See `http://persson.berkeley.edu/1B/sol174.pdf`