Worksheet 1 - Solutions

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1. Make the substitution $u = \sin(x)$. Then $du = \cos(x)dx$ and

$$\int \cos(x)e^{\sin(x)}dx = \int e^u du = e^u + C = e^{\sin(x)} + C$$

2. Make the substitution $x = -t^2$. Then $dx = -2tdt$, $tdt = -dx/2$ and

$$\int t^3 e^{-t^2} dt = \int t^2 e^{-t^2} \cdot tdt = \frac{1}{2}\int xe^x dx$$

Now use integration by parts (recall LIATE) with $u = x$ and $dv = e^x dx$. We get $v = e^x$ and

$$\frac{1}{2}\int xe^x dx = \frac{1}{2}(uv - \int vdu) = \frac{1}{2}(xe^x - \int e^x dx) = \frac{1}{2}(xe^x - e^x) + C$$

Returning to the variable t we obtain

$$\int t^3 e^{-t^2} dt = \frac{1}{2}(-t^2 e^{-t^2} - e^{-t^2}) + C$$

3. See the Solutions to Quiz 2.

4. Use integration by parts with $u = (1-x)^n$ and $dv = x^m dx$. We get $du = -n(1-x)^{n-1}$ and $v = x^{m+1}/(m+1)$. Therefore

$$\int x^m(1-x)^n dx = uv - \int vdu = \frac{1}{m+1}x^{m+1}(1-x)^n + \frac{n}{m+1}\int x^{m+1}(1-x)^{n-1}dx$$

$4'$. Applying the Fundamental Theorem of Calculus we get from the previous exercise that

$$\int_0^1 x^m(1-x)^n dx = \left[\frac{x^{m+1}(1-x)^n}{m+1}\right]_0^1 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx$$

Since the function $x^{m+1}(1-x)^n$ takes the value 0 at $x = 0$ and $x = 1$, we get

$$\int_0^1 x^m(1-x)^n dx = \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx$$

Iterating this we obtain

$$\begin{aligned}
\int_0^1 x^m(1-x)^n dx &= \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx \\
&= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\int_0^1 x^{m+2}(1-x)^{n-2}dx \\
&= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdot\frac{n-2}{m+3}\int_0^1 x^{m+3}(1-x)^{n-3}dx \\
&\vdots \\
&= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdots\frac{1}{m+n}\int_0^1 x^{m+n}(1-x)^0 dx \\
&= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdots\frac{1}{m+n}\cdot\frac{1}{m+n+1} = \frac{n!\cdot m!}{(n+m+1)!} \\
&= \frac{1}{(m+n+1)\binom{m+n}{m}}
\end{aligned}$$

(in the last expression, $\binom{m+n}{m}$ represents a binomial coefficient)

5. Using the half-angle formulas $\sin^2(t) = \frac{1-\cos(2t)}{2}$ and $\cos^2(t) = \frac{1+\cos(2t)}{2}$ we get

$$\begin{aligned}
\int_0^\pi \sin^2(t)\cos^4(t)dt &= \int_0^\pi \frac{1-\cos(2t)}{2}\left(\frac{1+\cos(2t)}{2}\right)^2 dt \\
&= \frac{1}{8}\int_0^\pi (1+\cos(2t)-\cos^2(2t)-\cos^3(2t))dt \\
&= \frac{1}{8}\int_0^\pi 1dt + \int_0^\pi \cos(2t)dt - \int_0^\pi \cos^2(2t)dt - \int_0^\pi \cos^3(2t)dt
\end{aligned}$$

We have

$$\int_0^\pi 1dt = 1\cdot(\pi-0) = \pi,$$

$$\int_0^\pi \cos(2t)dt = \left[\frac{\sin(2t)}{2}\right]_0^\pi = 0-0 = 0,$$

$$\int_0^\pi \cos^2(2t)dt = \int_0^\pi \frac{1+\cos(4t)}{2}dt = \frac{\pi}{2} + \left[\frac{\sin(4t)}{8}\right]_0^\pi = \frac{\pi}{2}.$$

To evaluate $\int_0^\pi \cos^3(2t)dt$ we use the fact that the exponent of cos is 3 (an odd number) and make the substitution $u = \sin(2t)$. We get $du = 2\cos(2t)dt$ and

$$\int \cos^3(2t)dt = \int (1-\sin^2(2t))\cos(2t)dt = \int (1-u^2)\frac{du}{2} = \frac{u}{2} - \frac{u^3}{6} = \frac{\sin(2t)}{2} - \frac{\sin(2t)^3}{6}$$

Using the Fundamental Theorem of Calculus we obtain

$$\int_0^\pi \cos^3(2t)dt = \left[\frac{\sin(2t)}{2} - \frac{\sin(2t)^3}{6}\right]_0^\pi = 0$$

Putting these together we obtain

$$\int_0^{\pi} \sin^2(t)\cos^4(t)dt = \frac{1}{8}(\pi + 0 - \frac{\pi}{2} - 0) = \frac{\pi}{16}$$

6. The exponent of sec is 2 (an even number), so we make the substitution $u = \tan(x)$. We obtain $du = \sec^2(x)dx$ and therefore

$$\int \sec^2(x)\tan(x)dx = \int udu = \frac{u^2}{2} + C = \frac{\tan^2(x)}{2} + C$$

7. Since $1 + \tan^2(x) = \sec^2(x)$ we get

$$\int (\tan^2(x) + \tan^4(x))dx = \int \tan^2(x)(1 + \tan^2(x))dx = \int \tan^2(x)\sec^2(x)dx$$

As in the previous example, the substitution $u = \tan(x)$ yields

$$\int \tan^2(x)\sec^2(x)dx = \int u^2du = \frac{u^3}{3} + C = \frac{\tan^3(x)}{3} + C$$

8. We will use the formula

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

that transforms a product of trigonometric functions into a sum. Letting $a = mx$ and $b = nx$ we get

$$\sin(mx)\cos(nx) = \frac{1}{2}(\sin(m+n)x + \sin(m-n)x)$$

hence

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = \frac{1}{2}(\int_{-\pi}^{\pi} \sin(m+n)xdx + \int_{-\pi}^{\pi} \sin(m-n)xdx)$$

Since m, n are positive, $m + n \neq 0$, so

$$\int_{-\pi}^{\pi} \sin(m+n)xdx = \left[\frac{-\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = \frac{-\cos(m+n)\pi + \cos(m+n)(-\pi)}{2} = 0$$

(for the last equality we used the fact that cos is an even function). If $m = n$ then $\sin(m-n)x = 0$. Otherwise $m - n \neq 0$ and therefore

$$\int_{-\pi}^{\pi} \sin(m-n)xdx = \left[\frac{-\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi} = \frac{-\cos(m-n)\pi + \cos(m-n)(-\pi)}{2} = 0$$

(again using the fact that cos is even) Putting these together we get

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = \frac{1}{2}(0+0) = 0$$