Worksheet 2 - Solutions

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1. The substitution $x = 10\tan\theta$ yields $dx = 10\sec^2\theta$ and

$$\int \frac{x^3}{\sqrt{x^2+100}}dx = \int \frac{10^3\tan^3\theta}{10\sec\theta}\cdot 10\sec^2\theta d\theta = 10^3\int \tan^3\theta\sec\theta d\theta$$

Now the substitution $u = \sec\theta$ yields $du = \tan\theta\sec\theta$ and

$$10^3\int \tan^3\theta\sec\theta = 10^3\int (u^2-1)du = 10^3\left(\frac{u^3}{3}-u\right)+C$$

We have $x = 10\tan\theta$, so

$$u = \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+(x/10)^2} = \frac{\sqrt{x^2+100}}{10}$$

and therefore

$$\int \frac{x^3}{\sqrt{x^2+100}}dx = 10^3\left(\frac{u^3}{3}-u\right)+C = \frac{(x^2+100)^{3/2}}{3} - 100\sqrt{x^2+100}+C$$

2. Completing the square we get $t^2-6t+13 = (t-3)^2+4$. The substitution $x = t-3$ yields

$$\int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{dx}{\sqrt{x^2+4}}$$

The substitution $x = 2\tan\theta$ yields

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{2\sec\theta}d\theta = \int \sec\theta d\theta = \ln|\sec\theta+\tan\theta|+C$$

We have $\tan\theta = x/2 = (t-3)/2$ and

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+(x/2)^2} = \sqrt{1+((t-3)/2)^2},$$

hence

$$\int \frac{dt}{\sqrt{t^2-6t+13}} = \ln\left|\frac{\sqrt{t^2-6t+13}}{2}+\frac{t-3}{2}\right|+C$$

3. Denote

$$I_n(x) = \int \frac{1}{(x^2+1)^n}dx$$

Our goal is to prove that

$$I_{n+1}(x) = \frac{1}{2n}\frac{x}{(x^2+1)^n} + \frac{2n-1}{2n}I_n(x).$$

Let's try to compute $I_n(x)$ using integration by parts. We write the integrand as a product in a trivial way:

$$\frac{1}{(x^2+1)^n}dx = \left(\frac{1}{(x^2+1)^n}\right)\cdot(1dx)$$

We integrate by parts letting $u = \dfrac{1}{(x^2+1)^n}$ and $dv = dx$. We get $v = x$ and

$$du = \frac{-2nx}{(x^2+1)^{n+1}}dx$$

so the formula

$$\int udv = uv - \int vdu$$

becomes

$$I_n(x) = \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2}{(x^2+1)^{n+1}}dx \qquad (*)$$

Writing $x^2 = (x^2+1) - 1$, we get

$$\frac{x^2}{(x^2+1)^{n+1}} = \frac{x^2+1}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}} = \frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}$$

hence

$$\int \frac{x^2}{(x^2+1)^{n+1}}dx = I_n(x) - I_{n+1}(x)$$

We get from (*) that

$$I_n(x) = \frac{x}{(x^2+1)^n} + 2n(I_n(x) - I_{n+1}(x)) = \frac{x}{(x^2+1)^n} + 2nI_n(x) - 2nI_{n+1}(x)$$

Moving $2nI_{n+1}(x)$ to the LHS and $I_n(x)$ to the RHS we obtain

$$2nI_{n+1}(x) = \frac{x}{(x^2+1)^n} + (2n-1)I_n(x)$$

Dividing by $2n$ we get the desired formula.

4. We first complete the square: $u^2 - 2u + 2 = (u-1)^2 + 1$ and make the substitution $x = u - 1$. We get

$$\int \frac{2u^3 - 5u^2 + 6u - 1}{(u^2 - 2u + 2)^2}du = \int \frac{2(x+1)^3 - 5(x+1)^2 + 6(x+1) - 1}{(x^2+1)^2}dx$$

$$= \int \frac{2 + 2x + x^2 + 2x^3}{(x^2+1)^2}dx$$

Now we use the method of partial fractions to write

$$\frac{2 + 2x + x^2 + 2x^3}{(x^2+1)^2} = \frac{a + bx}{x^2+1} + \frac{c + dx}{(x^2+1)^2}$$

Solving for a, b, c, d we get $a = 1$, $b = 2$, $c = 1$ and $d = 0$. Therefore

$$\int \frac{2 + 2x + x^2 + 2x^3}{(x^2 + 1)^2} dx = \int \frac{1}{x^2 + 1} dx + \int \frac{2x}{x^2 + 1} dx + \int \frac{1}{(x^2 + 1)^2} dx$$

The first two integrals are easy to compute: they are $\arctan(x)$ and $\ln |x^2+1|$ respectively. For the last one, we use the reduction formula from the previous exercise with $n = 1$ (alternatively you could use the substitution $x = \tan\theta$). We get

$$I_2(x) = \frac{x}{2(x^2 + 1)} + \frac{1}{2} I_1(x)$$

but $I_1(x) = \arctan(x)$, hence

$$I_2(x) = \frac{x}{2(x^2 + 1)} + \frac{\arctan(x)}{2}$$

Putting everything together we obtain

$$\int \frac{2 + 2x + x^2 + 2x^3}{(x^2 + 1)^2} dx = \arctan(x) + \ln |x^2 + 1| + \frac{x}{2(x^2 + 1)} + \frac{\arctan(x)}{2} + C$$

$$= \ln |x^2 + 1| + \frac{x}{2(x^2 + 1)} + \frac{3\arctan(x)}{2} + C$$

Going back to the variable u we obtain

$$\int \frac{2u^3 - 5u^2 + 6u - 1}{(u^2 - 2u + 2)^2} du = \ln |u^2 - 2u + 2| + \frac{u - 1}{2(u^2 - 2u + 2)} + \frac{3\arctan(u - 1)}{2} + C$$

5. Since the degree of the numerator is at least as large as the one of the denominator we first perform long division. We get

$$\frac{x^4 - 2x^3 - 14x^2 + 10}{x^3 - 3x^2 - 10x} = x + 1 + \frac{10 + 10x - x^2}{x^3 - 3x^2 - 10x}$$

We now factor the denominator to get $x^3 - 3x^2 - 10x = x(x + 2)(x - 5)$. Using the method of partial fractions, we write

$$\frac{10 + 10x - x^2}{x^3 - 3x^2 - 10x} = \frac{a}{x} + \frac{b}{x + 2} + \frac{c}{x - 5}$$

Solving for a, b, c yields $a = b = -1$ and $c = 1$. Therefore

$$\int \frac{10 + 10x - x^2}{x^3 - 3x^2 - 10x} dx = -\int \frac{dx}{x} - \int \frac{dx}{x + 2} + \int \frac{dx}{x - 5} = -\ln |x| - \ln |x+2| + \ln |x-5| + C$$

Adding the integral of $x + 1$ to this we obtain

$$\int \frac{x^4 - 2x^3 - 14x^2 + 10}{x^3 - 3x^2 - 10x} dx = \int (x+1) dx + \ln \left| \frac{x - 5}{x(x + 2)} \right| + C = \frac{x^2}{2} + x + \ln \left| \frac{x - 5}{x(x + 2)} \right| + C$$

6. The substitution $y = \sqrt{3x+2}$ yields $y^2 = 3x + 2$, $2ydy = 3dx$, hence

$$\int e^{\sqrt{3x+2}}dx = \int e^y \cdot \frac{2y}{3}dy = \frac{2}{3}\int ye^y dy$$

Integration by parts with $u = y$, $dv = e^y dy$ yields $v = e^y$ and

$$\int ye^y dy = ye^y - \int e^y dy = ye^y - e^y + C$$

Therefore

$$\int e^{\sqrt{3x+2}}dx = \frac{2}{3}(\sqrt{3x+2}e^{\sqrt{3x+2}} - e^{\sqrt{3x+2}}) + C$$

7. Make the substitution $y = x^2$, $dy = 2xdx$ to get

$$\int \frac{x}{\sqrt{3-x^4}}dx = \frac{1}{2}\int \frac{dy}{\sqrt{3-y^2}}$$

Now make the substitution $y = \sqrt{3}\sin\theta$ to get $dy = \sqrt{3}\cos\theta d\theta$ and

$$\int \frac{dy}{\sqrt{3-y^2}} = \int \frac{\sqrt{3}\cos\theta}{\sqrt{3}\cos\theta}d\theta = \theta + C = \arcsin(y/\sqrt{3}) + C$$

Going back to the variable x we get

$$\int \frac{x}{\sqrt{3-x^4}}dx = \frac{1}{2}\arcsin(x^2/\sqrt{3}) + C$$

8. Make the substitution $y = \sqrt[3]{x}$ to get $x = y^3$, $dx = 3y^2dy$ and

$$\int \frac{1}{x+\sqrt[3]{x}}dx = \int \frac{3y^2}{y^3+y}dy = 3\int \frac{y}{y^2+1}dy = \frac{3}{2}\ln|y^2+1| + C$$

Going back to the variable x we get

$$\int \frac{1}{x+\sqrt[3]{x}}dx = \frac{3}{2}\ln|x^{2/3}+1| + C$$

9. Recall LIATE: x is Algebraic, $\sin(x)^2\cos(x)$ is Trigonometric. Use integration by parts with $u = x$, $dv = \sin(x)^2\cos(x)dx$. The first step is to compute $v = \int dv = \int \sin(x)^2\cos(x)dx$. The exponent of cos is odd, so we make the substitution $w = \sin(x)$ to get $dw = \cos(x)dx$ and

$$v = \int w^2 dw = w^3/3 = \sin^3(x)/3$$

Going back to integration by parts we obtain

$$\int x\sin(x)^2\cos(x)dx = uv - \int vdu = \frac{x\sin^3(x)}{3} - \int \sin^3(x)/3dx$$

Now since the exponent of sin is odd, we make the substitution $w = \cos(x)$, $dw = -\sin(x)$ to get

$$\int \sin^3(x)/3dx = \frac{-1}{3}\int(1-w^2)dw = \frac{-w}{3} + \frac{w^3}{9} + C = \frac{-\cos(x)}{3} + \frac{\cos^3(x)}{9} + C$$

Combining this with the previous relations yields

$$\int x\sin(x)^2\cos(x)dx = \frac{x\sin^3(x)}{3} + \frac{\cos(x)}{3} - \frac{\cos^3(x)}{9} + C$$

10. The integrand has the form $f(x)\ln(x)$ where $f(x) = \frac{x}{\sqrt{x^2-1}}$ is an Algebraic function. We use integration by parts with $u = \ln(x)$, $dv = f(x)dx$, $du = \frac{1}{x}dx$. The first step is to integrate f:

$$v = \int f(x)dx = \int \frac{x}{\sqrt{x^2-1}}dx.$$

The substitution $x = \sec\theta$ yields $dx = \tan\theta\sec\theta d\theta$

$$v = \int \frac{\sec\theta}{\tan\theta}\tan\theta\sec\theta d\theta = \int \sec^2\theta d\theta = \tan\theta = \sqrt{x^2-1}$$

Going back to integration by parts we get

$$\int \frac{x\ln(x)}{\sqrt{x^2-1}}dx = \int udv = uv - \int vdu = \ln(x)\sqrt{x^2-1} - \int \frac{\sqrt{x^2-1}}{x}dx.$$

To integrate $\frac{\sqrt{x^2-1}}{x}$ we can use one of the following methods

Method 1 Use the substitution $x = \sec(\theta)$. This gives $dx = \tan(\theta)\sec(\theta)d\theta$, $\sqrt{x^2-1} = \tan(\theta)$, and therefore

$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x}dx &= \int \frac{\tan(\theta)}{\sec(\theta)}\tan(\theta)\sec(\theta)d\theta \\ &= \int \tan^2(\theta)d\theta = \int(\sec^2(\theta)-1)d\theta \\ &= \tan(\theta) - \theta + C = \sqrt{x^2-1} - \sec^{-1}(x) + C.\end{aligned}$$

Method 2 Use the substitution $y = \sqrt{x^2-1}$. This gives $y^2 = x^2-1$, and after differentiating $2ydy = 2xdx$. Therefore

$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x}dx &= \int \frac{y}{x^2}xdx = \int \frac{y}{y^2+1}ydy \\ &= \int(1 - \frac{1}{y^2+1})dy = y - \tan^{-1}(y) + C = \sqrt{x^2-1} - \tan^{-1}(\sqrt{x^2-1}) + C.\end{aligned}$$

Putting together all of the above, we get that

$$\int \frac{x\ln(x)}{\sqrt{x^2-1}}dx = \ln(x)\sqrt{x^2-1} - \sqrt{x^2-1} + \tan^{-1}(\sqrt{x^2-1}) + C.$$