Worksheet 4 - Solutions

Claudiu Raicu

February 17, 2010

1. We have
$$y'(x) = \frac{1}{\frac{e^x+1}{e^x-1}} \cdot \frac{e^x(e^x-1)-(e^x+1)e^x}{(e^x-1)^2} = \frac{-2e^x}{e^{2x}-1}$$
hence
$$1+(y'(x))^2 = 1+\frac{4e^{2x}}{(e^{2x}-1)^2} = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}.$$
We get
$$L = \int_a^b \sqrt{1+(y'(x))^2}dx = \int_a^b \frac{e^{2x}+1}{e^{2x}-1}dx = \int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$$
(where the last equality comes from dividing both numerator and denominator by e^x). Now since
$$\cosh(x) = \frac{e^x+e^{-x}}{2},\ \sinh(x) = \frac{e^x-e^{-x}}{2},$$
we see that
$$L = \int_a^b \frac{\cosh(x)}{\sinh(x)}dx$$
Making the substitution $u = \sinh(x)$ we get $du = \cosh(x)dx$ and therefore
$$L = \int_{\sinh(a)}^{\sinh(b)} \frac{du}{u} = [\ln|u|]_{\sinh(a)}^{\sinh(b)} = \ln\left|\frac{\sinh(b)}{\sinh(a)}\right| = \ln\left|\frac{e^b-e^{-b}}{e^a-e^{-a}}\right|$$

2. We have $y'(x) = \sqrt{x^3-1}$, so $1+(y'(x))^2 = x^3$. We get
$$L = \int_1^4 \sqrt{x^3}dx = \left[\frac{2}{5}x^{5/2}\right]_1^4 = \frac{2}{5}(32-1) = \frac{62}{5}$$

3. Let s denote the arc length function. We have
$$s(x) = \int_0^x \sqrt{1+(y'(t))^2}dt.$$
Since
$$y'(t) = \frac{1}{\sqrt{1-t^2}} - \frac{t}{\sqrt{1-t^2}},$$
we get
$$\sqrt{1+(y'(t))^2} = \sqrt{1+\frac{(1-t)^2}{1-t^2}} = \sqrt{1+\frac{1-t}{1+t}} = \sqrt{\frac{2}{1+t}}.$$
It follows that
$$s(x) = \int_0^x \sqrt{2}(1+t)^{-1/2}dt = \sqrt{2}(2(1+t)^{1/2})|_0^x = 2\sqrt{2}(\sqrt{1+x}-1).$$

4. Let s denote the arc length function. We have

$$s(x) = \int_1^x \sqrt{1+(y'(t))^2}dt.$$

Since

$$y'(t) = 2 \cdot \frac{3}{2} \cdot t^{1/2} = 3\sqrt{t},$$

we get

$$\sqrt{1+(y'(t))^2} = \sqrt{1+9t}.$$

It follows that

$$s(x) = \int_1^x \sqrt{1+9t}dt = \left[\frac{2}{3} \cdot \frac{(1+9t)^{3/2}}{9}\right]_1^x = \frac{2(1+9x)^{3/2}}{27} - \frac{2(10)^{3/2}}{27}$$

5. For rotation about the x-axis, the surface area formula is

$$S = \int 2\pi y ds = 2\pi \int_0^1 y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

We have

$$\frac{dy}{dx} = \pi\cos(\pi x)$$

We get

$$S = 2\pi \int_0^1 \sin(\pi x)\sqrt{1+(\pi\cos(\pi x))^2}dx$$

We make the substitution $u = \pi\cos(\pi x)$ to get $du = -\pi^2\sin(\pi x)dx$, or $\sin(\pi x)dx = -du/\pi^2$ and

$$S = 2\pi \int_\pi^{-\pi} \frac{-\sqrt{1+u^2}}{\pi^2}du = \frac{2}{\pi}\int_{-\pi}^{\pi} \sqrt{1+u^2}du$$

(notice that we got rid of the '–' sign by interchanging the limits of integration). Since $\sqrt{1+u^2}$ is an even function, we get

$$S = \frac{4}{\pi}\int_0^\pi \sqrt{1+u^2}du$$

To compute $\int \sqrt{1+u^2}du$ we use the substitution $u = \tan\theta$, $du = \sec^2\theta d\theta$. We get

$$\int \sqrt{1+u^2}du = \int \sec^3\theta d\theta$$

To compute this, use integration by parts $v = \sec\theta$, $dw = \sec^2\theta$, which gives $w = \tan\theta$ and $dv = \tan\theta\sec\theta d\theta$. We get

$$\begin{aligned}
\int \sec^3\theta d\theta &= vw - \int w dv = \sec\theta\tan\theta - \int \tan^2\sec\theta d\theta \\
&= \sec\theta\tan\theta - \int(\sec^2\theta - 1)\sec\theta d\theta = \sec\theta\tan\theta - \int\sec^3\theta d\theta + \int\sec\theta d\theta \\
&= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int\sec^3\theta d\theta
\end{aligned}$$

Moving $\int \sec^3\theta d\theta$ to the LHS and dividing by 2 we get

$$\int \sec^3\theta d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)$$

Going back to the variable u, we have

$$\int \sqrt{1+u^2}du = \frac{1}{2}(u\sqrt{1+u^2} + \ln|u+\sqrt{1+u^2}|)$$

It follows that

$$\begin{aligned} S &= \frac{4}{\pi}\int_0^{\pi} \sqrt{1+u^2}du = \frac{4}{\pi}\cdot\frac{1}{2}[u\sqrt{1+u^2} + \ln|u+\sqrt{1+u^2}|]|_0^{\pi} \\ &= \frac{2}{\pi}(\pi\sqrt{1+\pi^2} + \ln|\pi+\sqrt{1+\pi^2}|) \end{aligned}$$

6. For rotation about the y-axis, the surface area formula is

$$\begin{aligned} S = \int 2\pi x ds &= 2\pi\int_0^1 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\ &= 2\pi\int_0^1 x\sqrt{1+4x^2}dx. \end{aligned}$$

The substitution $u = 4x^2$ yields $du = 8xdx$, hence

$$S = \frac{2\pi}{8}\int_0^4 \sqrt{1+u}du = \frac{\pi}{4}\cdot\frac{2}{3}(1+u)^{3/2}|_0^4 = \frac{\pi}{6}(5^{3/2}-1).$$

Notice that instead of the substitution $u = 4x^2$ you could have used the trig substitution $x = \frac{1}{2}\tan\theta$.

7. For rotation about the y-axis, the surface area formula is

$$S = \int 2\pi x ds = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

We have

$$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$$

so

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{x^2}{4} - 2\frac{x}{2}\frac{1}{2x} + \frac{1}{4x^2}\right) = \left(\frac{x}{2}+\frac{1}{2x}\right)^2$$

We get

$$S = 2\pi\int_1^2 x\left(\frac{x}{2}+\frac{1}{2x}\right)dx = 2\pi\left[\frac{x^3}{6}+\frac{x}{2}\right]_1^2 = 2\pi\left(\frac{2^3-1^3}{6}+\frac{1}{2}\right) = \frac{10\pi}{3}$$

8. For rotation about the x-axis, the surface area formula is

$$S = \int 2\pi y ds = 2\pi\int_0^{\infty} y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

We have $y'(x) = -e^{-x}$, so

$$S = 2\pi \int_0^{\infty} e^{-x}\sqrt{1+e^{-2x}}dx$$

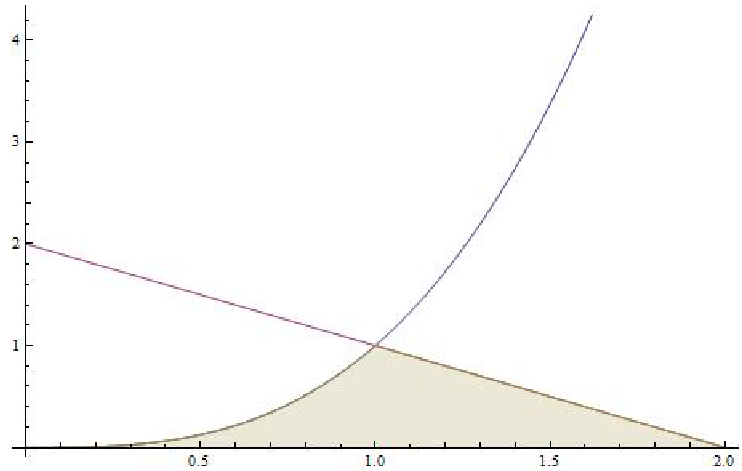
The substitution $u = e^{-x}$, $du = -e^{-x}dx$ yields $e^{-x}dx = -du$ and (since $e^{-0} = 1$, $e^{-\infty} = 0$)

$$\begin{aligned} S &= 2\pi \int_1^0 -\sqrt{1+u^2}du = 2\pi \int_0^1 \sqrt{1+u^2}du \\ &= 2\pi\frac{1}{2}[u\sqrt{1+u^2} + \ln|u+\sqrt{1+u^2}|]|_0^1 = \pi(\sqrt{2} + \ln|1+\sqrt{2}|) \end{aligned}$$

(here we used the calculation of $\int \sqrt{u^2+1}du$ from problem 5).

9. The region bounded by the curves $y = x^3, x+y = 2, y = 0$ is the region under the graph of f (see the figure below), where

$$f(x) = \begin{cases} x^3, & 0 \le x \le 1; \\ 2-x, & 1 \le x \le 2. \end{cases}$$



The coordinates $(\overline{x}, \overline{y})$ of the centroid are given by

$$\overline{x} = \frac{1}{A}\int_0^2 xf(x)dx,\ \overline{y} = \frac{1}{A}\int_0^2 \frac{1}{2}f(x)^2dx,$$

where A is the area below the graph of f,

$$A = \int_0^2 f(x)dx = \int_0^1 x^3dx + \int_1^2 (2-x)dx = \frac{x^4}{4}|_0^1 - \frac{(2-x)^2}{2}|_1^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

We have

$$\int_0^2 xf(x)dx = \int_0^1 x^4dx + \int_1^2 (2x-x^2)dx = \frac{x^5}{5}|_0^1 + (x^2 - \frac{x^3}{3})|_1^2 = \frac{1}{5} + \frac{4}{3} - \frac{2}{3} = \frac{13}{15},$$

and

$$\int_0^2 f(x)^2dx = \int_0^1 x^6dx + \int_1^2 (2-x)^2dx = \frac{x^7}{7}|_0^1 - \frac{(2-x)^3}{3}|_1^2 = \frac{1}{7} + \frac{1}{3} = \frac{10}{21}.$$

It follows that

$$\overline{x} = \frac{4}{3} \cdot \frac{13}{15} = \frac{52}{45}$$

and

$$\overline{y} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{10}{21} = \frac{20}{63}.$$

10. Assume the coordinate axes are such that the origin lives at the bottom of one end of the tank. Then at level y, the length of the cross section of the end of the tank is $l(y) = 2\sqrt{2y}$. We get

$$F = \rho g \int_0^8 2\sqrt{2y}(8-y)dy = \rho g \int_0^8 (2^{9/2}y^{1/2} - 2^{3/2}y^{3/2})dy$$
$$= \rho g \left[\frac{2^{9/2}y^{3/2}}{3/2} - \frac{2^{3/2}y^{5/2}}{5/2}\right]_0^8 = \rho g \left(\frac{2^{10}}{3} - 2^{10}5\right) = \rho g \frac{2^{11}}{15}$$

11. Such a cone is obtained by rotating a right triangle with sides $h, r, \sqrt{h^2+r^2}$ about an axis containing the side of length h. Assuming the axis is the x-axis, and that the origin is situated at the vertex of the triangle which is the intersection of the hypothenuse with the side of length h, we get that the coordinates of the centroid are

$$\overline{x} = \frac{0+h+h}{3} = \frac{2h}{3},\ \overline{y} = \frac{0+0+r}{3} = \frac{r}{3}$$

The centroid then moves on a circle of radius $r/3$. Using the fact that the area of the triangle is $rh/2$, it follows from the Theorem of Pappus that

$$V = \frac{2\pi r}{3} \cdot \frac{rh}{2} = \frac{\pi r^2 h}{3}$$