Worksheet 7 - Solutions

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1. We use the Root Test (the Ratio Test also works - try it!):

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{(-1)^n x^n}{n+1}\right|} = \frac{|x|}{\sqrt[n]{n+1}}$$

Since $\lim_{n\to\infty} \sqrt[n]{n+1} = 1$, we get

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = |x|.$$

The Radius of Convergence is then given by

$$|x| < 1, \text{ so } R = 1.$$

To get the interval of convergence, we have to test $x = 1$ and $x = -1$. When $x = 1$, we get the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

which is convergent by the Alternating Series Test (you'll need to fill in the details here). When $x = -1$, we get the series

$$\sum_{n=0}^{\infty} \frac{1}{n+1}$$

which is divergent being the Harmonic Series. Summing up, the Interval of Convergence is $I = (-1, 1]$ and the Radius of Convergence is $R = 1$.

2. We use the Ratio Test (the Root Test also works - try it!):

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(10x)^{n+1}}{(n+1)^3}}{\frac{(10x)^n}{n^3}}\right| = |10x|\frac{n^3}{(n+1)^3}$$

Since $\lim_{n\to\infty} n^3/(n+1)^3 = 1$, we get

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 10|x|.$$

The Radius of Convergence is then given by

$$10|x| < 1, \text{ so } R = \frac{1}{10} = 0.1$$

To get the interval of convergence, we have to test $x = 1/10$ and $x = -1/10$. When $x = 1/10$, we get the series

$$\sum_{n=0}^{\infty} \frac{1}{n^3}$$

which is convergent by the p-series test. When $x = -1/10$, we get the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3}$$

which is convergent by the Alternating Series Test. In conclusion, the Interval of Convergence is $I = [-1/10, 1/10]$ and the Radius of Convergence is $R = 1/10$.

3. We use the Root Test:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{(x-2)^n}{n^n}\right|} = \frac{|x-2|}{n}$$

We then have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0,$$

so the Radius of Convergence is ∞ and the Interval of Convergence is $(-\infty, \infty)$.

4. We use the Root Test (the Ratio Test also works - try it!):

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{(-1)^n x^n}{4^n \ln n}\right|} = \frac{|x|}{4\sqrt[n]{\ln\ n}}$$

Since $1 \le \ln\ n \le n$, taking n-th roots we obtain $1 \le \sqrt[n]{\ln\ n} \le \sqrt[n]{n}$. We know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$, so that $1 \le \lim_{n\to\infty} \sqrt[n]{\ln\ n} \le 1$. This implies

$$\lim_{n\to\infty} \sqrt[n]{\ln\ n} = 1$$

and therefore

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|x|}{4}.$$

The Radius of Convergence is then given by

$$\frac{|x|}{4} < 1 \Leftrightarrow |x| < 4, \text{ so } R = 4.$$

To get the interval of convergence, we have to test $x = 4$ and $x = -4$. When $x = 4$, we get the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln\ n}$$

which is convergent by the Alternating Series Test (you'll need to fill in the details here). When $x = -4$, we get the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln\ n}$$

which is divergent by the Comparison Test (you'll need to fill in the details here - use the fact that $1/(\ln\ n) > 1/n$, and that the Harmonic Series is convergent). Summing up, the Interval of Convergence is $I = (-4, 4]$ and the Radius of Convergence is $R = 4$.

5. We use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(n+1)^2(2x-3)^{n+1}}{2\cdot 4\cdots(2n+2)}}{\dfrac{n^2(2x-3)^n}{2\cdot 4\cdots(2n)}}\right| = \frac{|2x-3|}{2n+2}\cdot\frac{(n+1)^2}{n^2}$$

Since $\lim_{n\to\infty}(n+1)^2/n^2 = 1$ and $\lim_{n\to\infty}|2x-3|/(2n+2) = 0$, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.$$

The Radius of Convergence is therefore ∞ and the Interval of Convergence is $(-\infty, \infty)$.

6. We use the Root Test (the Ratio Test also works - try it!):

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{(4x+1)^n}{n^2}\right|} = \frac{|4x+1|}{\sqrt[n]{n^2}}$$

Since $\lim_{n\to\infty}\sqrt[n]{n} = 1$, we get $\lim_{n\to\infty}\sqrt[n]{n^2} = 1^2 = 1$, hence

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = |4x+1|.$$

The Radius of Convergence is then given by

$$|4x+1| < 1 \Leftrightarrow |x-(-1/4)| < 1/4, \text{ so } R = 1/4.$$

Note that we have $a = -1/4$. To get the interval of convergence, we have to test $x = a + R = 0$ and $x = a - R = -1/2$. When $x = 0$, we get the series

$$\sum_{n=0}^{\infty}\frac{1}{n^2}$$

which is convergent by the p-series test. When $x = -1/2$, we get the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$$

which is convergent by the Alternating Series Test. In conclusion, the Interval of Convergence is $I = [-1/2, 0]$ and the Radius of Convergence is $R = 1/4$.

7. We use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = (n+1)|2x-1|.$$

We get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty \text{ for } x \neq 1/2.$$

The Radius of Convergence is therefore 0 and the Interval of Convergence reduces to the single point $1/2$: $I = [1/2, 1/2]$.

8. We apply the Root Test (the Ratio Test also works - try it!), using the fact that

$$\lim_{n\to\infty} \sqrt[n]{\ln\ n} = 1 \text{ and } \lim_{n\to\infty} \sqrt[n]{n} = 1$$

(see Exercise 4). We have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{x^{2n}}{n(\ln n)^2}\right|} = \frac{|x|^2}{1 \cdot 1^2} = |x|^2.$$

The Radius of Convergence is then given by

$$|x|^2 < 1 \Leftrightarrow |x| < 1, \text{ so } R = 1.$$

To get the interval of convergence, we have to test $x = 1$ and $x = -1$. When $x = 1$, we get the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln\ n)^2}$$

which is convergent by the Integral Test. More explicitly, the sequence $1/(n(\ln\ n)^2)$ is decreasing and positive, so we can apply the Integral Test to conclude that the convergence of the series is equivalent to the convergence of the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln\ x)^2}$$

The substitution $u = \ln\ x$, $du = dx/x$ yields

$$\int \frac{dx}{x(\ln\ x)^2} = \int \frac{du}{u^2} = \frac{-1}{u} + C = \frac{-1}{\ln(x)} + C$$

so

$$\int_2^{\infty} \frac{dx}{x(\ln\ x)^2} = \lim_{t\to\infty} \left[\frac{-1}{\ln(x)}\right]_2^t = \frac{1}{\ln(2)}$$

and we see that the integral is convergent, hence so is our original series.

When $x = -1$, we get the same series as for $x = 1$, because $(-1)^{2n} = 1^{2n}$, so we also have convergence in this case. In conclusion, the Interval of Convergence is $I = [-1, 1]$ and the Radius of Convergence is $R = 1$.

9. We have

$$\frac{3}{1 - x^4} = 3\sum_{n=0}^{\infty}(x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}.$$

Since the interval of convergence for the geometric series is determined by the inequality $|x| < 1$, our series converges if and only if $|x|^4 < 1$, or equivalently $|x| < 1$. Therefore $ROC = 1$ and $IOC = (-1, 1)$. (Use the standard method to verify these assertions)

10. We have

$$\frac{1 + x}{1 - x} = \frac{-(1 - x) + 2}{1 - x} = -1 + \frac{2}{1 - x} = -1 + \sum_{n=0}^{\infty} 2x^n.$$

Since the geometric series has $ROC = 1$ and $IOC = (-1, 1)$, the same is true about our series. (Use the standard method to verify these assertions)

11. Using partial fractions we get

$$\frac{x+2}{2x^2-x-1} = \frac{-1}{1-x} - \frac{1}{1+2x} = -\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}((-1)^{n+1}2^n - 1)x^n$$

Our series is therefore the sum between a series with $IOC = (-1, 1)$ and one with $IOC = (-1/2, 1/2)$. It follows that it has $IOC = (-1/2, 1/2)$. (Use the standard method to verify these assertions)

12. We have

$$\begin{aligned} f(x) &= \frac{x^2}{2} \cdot \left(\frac{1}{1-2x}\right)' = \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty}(2x)^n\right)' \\ &= \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} 2^n n x^{n-1}\right) + C = \sum_{n=0}^{\infty} 2^{n-1} n x^{n+1} + C \end{aligned}$$

To determine C, we plug in $x = 0$ and get $f(0) = 0 + C$, i.e. $C = 0$, so

$$f(x) = \sum_{n=0}^{\infty} 2^{n-1} n x^{n+1}.$$

The series expansion for $1/(1-2x)$ has $ROC = 1/2$, and we know that the radius of convergence doesn't change when we take derivatives or integrate. Therefore our series also has $ROC = 1/2$. However, the interval of convergence is not preserved, so we need to test for the end points. When $x = 1/2$ we get

$$\sum_{n=0}^{\infty} 2^{n-1} n \frac{1}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{n}{4}$$

which is clearly divergent (use the Test for Divergence, $n/4$ does not go to 0 as n goes to infinity). Similarly, when $x = -1/2$ we get

$$\sum_{n=0}^{\infty} 2^{n-1} n \frac{1}{(-2)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n}{4}$$

and by the same reasoning as before we obtain the divergence of the series. It follows that $IOC = (-1/2, 1/2)$. (Use the standard method to verify these assertions)

13. We have

$$f'(x) = \frac{1}{1+(x/3)^2} \cdot \frac{1}{3} = \frac{1}{3} \cdot \frac{1}{1-(-x^2/9)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x^2}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}} x^{2n}$$

It follows that

$$f(x) = \int f'(x)dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}} \cdot \frac{x^{2n+1}}{2n+1} + C.$$

To determine C, we plug in $x = 0$ and get $f(0) = 0 + C$, i.e. $C = 0$. It follows that

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{3^{2n+1}(2n+1)}\right) x^{2n+1}$$

The radius of convergence for $f'(x)$ is given by $(x/3)^2 < 1$, or equivalently $|x| < 3$, so $ROC = 3$. Therefore the one for $f(x)$ has to be the same, and to determine IOC we need to test the endpoints $x = \pm 3$. For $x = 3$ we get

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{3^{2n+1}(2n+1)} \right) \cdot 3^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

which is convergent by the Alternating Series Test. When $x = -3$ we get

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{3^{2n+1}(2n+1)} \right) \cdot (-3)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

which is again convergent by the Alternating Series Test. It follows that $IOC = [-3, 3]$. (Use the standard method to verify these assertions)

14. We have

$$\frac{x}{x^2+16} = \frac{x}{16} \cdot \frac{1}{1-(-x^2/16)} = \frac{x}{16} \sum_{n=0}^{\infty} \left(\frac{-x^2}{16} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{16^{n+1}} x^{2n+1}$$

The IOC for this series is given by the inequality $|-x^2/16| < 1$, or equivalently $|x| < 4$, so $IOC = (-4, 4)$ and $ROC = 4$. (Use the standard method to verify these assertions)

15. We have

$$f'(x) = \frac{2x}{x^2+4} = \frac{2x}{4} \cdot \frac{1}{1 - \dfrac{-x^2}{4}} = \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{-x^2}{4} \right)^n$$
$$= \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} x^{2n+1}$$

It follows that

$$f(x) = \int f'(x)dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \cdot \frac{x^{2n+2}}{2n+2} + C = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2}(n+1)} x^{2n+2} + C.$$

To determine C, we plug in $x = 0$ and get $f(0) = 0 + C$, i.e. $C = \ln(4)$. It follows that

$$f(x) = \ln(4) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n} \cdot n} x^{2n}$$

The radius of convergence for $f'(x)$ is given by $|-x^2/4| < 1$, or equivalently $|x| < 2$, so $ROC = 2$. Therefore the one for $f(x)$ has to be the same, and to determine IOC we need to test the endpoints $x = \pm 2$. At both $x = 2$ and $x = -2$ we get the series

$$\ln(4) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n} \cdot n} 2^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which is convergent by the Alternating Series Test. It follows that $IOC = [-2, 2]$. (Use the standard method to verify these assertions)

16. (a) Taking the derivative in the equality

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

we get

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}.$$

(b) Multiplying by x both sides of the previous equality, we get

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n.$$

(c) Plugging in $x = 1/2$ in part (b), we obtain

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2.$$

(d) Taking the derivative of the equality in part (a), we get

$$\frac{2}{(1-x)^3} = \sum_{n=1}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}.$$

Multiplying both sides by x^2, we obtain

$$\frac{2x^2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^n.$$

(e) Plugging in $x = 1/2$ in part (d) yields

$$\sum_{n=1}^{\infty} \frac{n^2-n}{2^n} = \sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \frac{2(1/2)^2}{(1-1/2)^3} = 4$$

(f) Adding (c) and (e) we obtain

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n}{2^n} + \sum_{n=1}^{\infty} \frac{n^2-n}{2^n} = 2 + 4 = 6.$$