Worksheet 8 - Solutions

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1. We have
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
so replacing x by $2x$ we get
$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$
It follows that
$$f(x) = \sum_{n=0}^{\infty} \frac{2^n + 1}{n!} x^n.$$

2. The Taylor series for $\tan^{-1}(x)$ is
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$
so replacing x by x^3 we get
$$\tan^{-1}(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$$
It follows that
$$f(x) = x^2 \tan^{-1}(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{6n+5}$$

3. Using the Binomial Expansion, we get
$$\begin{aligned}\frac{x}{\sqrt{x^2+4}} &= \frac{x}{2} \cdot \left(1 + \frac{x^2}{4}\right)^{-1/2} = \frac{x}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x^2}{4}\right)^n \\ &= \frac{x}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{x^{2n}}{2^{2n}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{x^{2n+1}}{2^{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{3n+1} n!} x^{2n+1}\end{aligned}$$

4. Using the half angle formula

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

and the binomial expansion for $\cos(2x)$

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

we obtain

$$f(x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1}}{(2n)!} x^{2n}$$

5. We have

$$\begin{aligned}\sin^{-1} x &= \int (\sin^{-1} x)' dx = \int \frac{1}{\sqrt{1-x^2}} dx = \int (1-x^2)^{-1/2} dx \\ &= \int \left(\sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n \right) dx = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \frac{x^{2n+1}}{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n! (2n+1)} x^{2n+1}\end{aligned}$$

6. We have

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots$$

so that

$$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} \right) + \text{higher order terms} = \frac{x^3}{3} + \text{higher order terms}$$

It follows that

$$\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x \to 0} \frac{x^3/3}{x^3} = \frac{1}{3}.$$

7. We have

$$\cos\ x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$$

so that

$$1 - \cos\ x = 1 - \left(1 - \frac{x^2}{2} \right) + \text{higher order terms} = \frac{x^2}{2} + \text{higher order terms.}$$

We also have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$$

so that

$$1 + x - e^x = 1 + x - \left(1 + x + \frac{x^2}{2} \right) + \text{higher order terms} = -\frac{x^2}{2} + \text{higher order terms.}$$

It follows that

$$\lim_{x \to 0} \frac{1 - \cos\ x}{1 + x - e^x} = \lim_{x \to 0} \frac{x^2/2}{-x^2/2} = -1$$

8. We have

$$\lim_{x\to 0}\frac{\tan\ x - x}{x^3} = \lim_{x\to 0}\frac{T_3(x)}{x^3}$$

where $T_3(x)$ is the 3rd Taylor polynomial associated to the function $f(x) = \tan\ x - x$ at $a = 0$. We need to compute the first three derivatives of f:

$$f'(x) = \sec^2 x - 1,$$

$$f''(x) = 2\sec\ x\cdot\sec\ x\tan\ x = 2\sec^2 x\tan\ x,$$
$$f'''(x) = 4\sec^2 x\tan^2 x + 2\sec^4 x$$

We get

$$f(0) = 0,\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = 2$$

so that

$$T_3(x) = \sum_{i=0}^{3}\frac{f^{(i)}(x)}{i!}x^i = \frac{2}{3!}x^3 = \frac{x^3}{3}.$$

We conclude that

$$\lim_{x\to 0}\frac{\tan\ x - x}{x^3} = \lim_{x\to 0}\frac{T_3(x)}{x^3} = \lim_{x\to 0}\frac{x^3/3}{x^3} = \frac{1}{3}$$

9. We have

$$f(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

The formula for $f^{(n)}(x)$ depends only on the parity of n, hence we can find a uniform bound for all the derivatives of $\sinh\ x$ on any interval. Taylor's inequality then proves that

$$f(x) = \lim_{n\to\infty} T_n(x) \text{ for all } x$$

i.e. the Maclaurin series for $f(x)$ represents $f(x)$ for all values of x.

10. We have

$$\sin\ x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

We think of x as a power series, and use division of power series to calculate y. The ratio between x and the leading term of $\sin\ x$ is 1, so the Division Algorithm yields 1 as the first term in the series of $x/\sin\ x$. Now we subtract $1\cdot\sin\ x$ from x, and proceed with the algorithm:

$$x - 1\cdot\sin\ x = x^3/6 - x^5/120 + \cdots$$

The ratio between the leading term of the new series and the leading term of $\sin\ x$ is $x^3/6x = x^2/6$, so the next term in the expansion of $x/\sin\ x$ is $x^2/6$. Continuing

$$(x^3/6 - x^5/120 + \cdots) - \frac{x^2}{6}\cdot\sin\ x = \frac{-x^5}{120} + \frac{x^5}{6\cdot 6} + \cdots = \frac{7x^5}{360} + \cdots$$

and we get the third term in the expansion of $x/\sin\ x$ as $7x^5/360x = 7x^4/360$. In conclusion,

$$\frac{x}{\sin\ x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \cdots$$

11. We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

so we can use multiplication of power series to get that

$$\begin{aligned} e^x \ln(1-x) &= -(1 + x + x^2/2! + \cdots)(x + x^2/2 + x^3/3) \\ &= -(x + (1+1/2)x^2 + (1/3 + 1/2 + 1/2)x^3 + \cdots) = -x - \frac{3}{2}x^2 - \frac{4}{3}x^2 - \cdots \end{aligned}$$

12. We have

$$\begin{gathered} f(1) = \ln(3), \\ f'(x) = \frac{2}{1+2x}, \quad f'(1) = \frac{2}{3}, \\ f''(x) = \frac{-4}{(1+2x)^2}, \quad f''(1) = \frac{-4}{9}, \\ f'''(x) = \frac{16}{(1+2x)^3}, \quad f'''(1) = \frac{16}{27}, \\ f^{(4)}(x) = \frac{-96}{(1+2x)^4}. \end{gathered}$$

It follows that

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(x)}{n!}(x-1)^n = \ln(3) + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3.$$

We estimate the accuracy of the approximation $f(x) \approx T_3(x)$ via Taylor's inequality:

$$|f(x) - T_3(x)| \leq \frac{M}{4!}|x-1|^4,$$

where M is such that

$$|f^{(4)}(x)| \leq M \text{ for } 0.5 \leq x \leq 1.5$$

We have

$$|f^{(4)}(x)| = \frac{96}{(1+2x)^4}$$

which is a decreasing function on the interval $[0.5, 1.5]$ (take the derivative), so its maximal value is attained at the left endpoint of the interval

$$|f^{(4)}(x)| \leq \frac{96}{(1+2\cdot 0.5)^4} = \frac{96}{2^4} = 6,$$

and therefore we can take $M = 6$. Also $|x-1| \leq 0.5 = 1/2$, hence $|x-1|^4 \leq 1/16$. We get

$$|f(x) - T_3(x)| \leq \frac{6}{4!} \cdot \frac{1}{16} = \frac{1}{64}, \text{ for } 0.5 \leq x \leq 1.5$$

13. We have $f(0) = 0$,

$$f'(x) = x\cos(x) + \sin(x), \text{ so } f'(0) = 0, \ f''(x) = -x\sin(x) + 2\cos(x), \text{ so } f''(0) = 2$$
$$f'''(x) = -(x\cos(x) + \sin(x)) - 2\sin(x), \text{ so } f'''(0) = 0 \text{ and}$$
$$f^{(4)}(x) = x\sin(x) - 4\cos(x), \text{ so } f^{(4)}(0) = -4$$

It follows that

$$T_4(x) = \frac{2}{2!}x^2 - \frac{4}{4!}x^4 = x^2 - \frac{x^4}{6}.$$

Taylor's inequality tells us that

$$|R_4(x)| = |f(x) - T_4(x)| \le \frac{M}{5!}|x|^5$$

where M is an upper bound for $f^{(5)}(x)$ on the given interval $[-1, 1]$. We have

$$|f^{(5)}(x)| = |x\cos(x) + 5\sin(x)| \le |x| \cdot |\cos(x)| + 5|\sin(x)| \le 1 \cdot 1 + 5 = 6$$

so we can take $M = 6$. Taylor's inequality becomes

$$|R_4(x)| \le \frac{6}{120}|x|^5$$

Since $|x| \le 1$ for $x \in [-1, 1]$, we get

$$|R_4(x)| \le \frac{1}{20}|x|^5 \le \frac{1}{20} = 0.05$$

so the approximation is accurate to within 0.05.

14. We have

$$\cos\ x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!},$$

which is an alternating series whose first terms are $1, -x^2/2, x^4/24, -x^6/720$. Recall that for a series

$$s = \sum_{n=0}^{\infty}(-1)^n a_n,$$

with partial sums

$$s_k = \sum_{n=0}^{k}(-1)^n a_n,$$

the Alternating Series Estimation Theorem yields

$$|s - s_k| \le |a_{k+1}|.$$

In our case $a_n = \dfrac{x^{2n}}{(2n)!}$, hence

$$|\cos\ x - (1 - \frac{x^2}{2} + \frac{x^4}{24})| < \frac{|x|^6}{720}.$$

To make the error smaller than 0.005 it suffices to have

$$\frac{|x|^6}{720} < 0.005 \Leftrightarrow |x| < \sqrt[6]{3.6} \approx 1.24.$$