Worksheet 9 - Solutions

Claudiu Raicu

April 12, 2010

1. Let's first check the initial condition:

$$y(0) = \sin(0)\cos(0) - \cos(0) = 0 \cdot 1 - 1 = -1.$$

To see that y is a solution of the differential equation, we calculate

$$y'(x) = \sin(x) \cdot (-\sin(x)) + \cos(x) \cdot \cos(x) + \sin(x) = -\sin^2(x) + \cos^2(x) + \sin(x),$$

$$\tan(x) \cdot y = \frac{\sin(x)}{\cos(x)} \cdot (\sin(x)\cos(x) - \cos(x)) = \sin^2(x) - \sin(x)$$

from which it follows that

$$y' + (\tan\ x)y = -\sin^2(x) + \cos^2(x) + \sin(x) + \sin^2(x) - \sin(x) = \cos^2(x).$$

We thus see that y is a solution of the given initial-value problem.

2. (a) We have $y'(t) = -k\sin(kt)$ and $y''(t) = -k^2\cos(kt) = -k^2 y(t)$. Therefore, in order for y to satisfy the differential equation

$$4y'' = -25y, \text{ which is equivalent to } y'' = \frac{-25}{4}y$$

we must have

$$-k^2 = \frac{-25}{4} \Leftrightarrow k^2 = (5/2)^2 \Leftrightarrow k = 5/2 \text{ or } k = -5/2.$$

(b) If $y = A\sin\ kt + B\cos\ kt$ then $y' = Ak\cos(kt) - Bk\sin(kt)$ and

$$y''(t) = -Ak^2\sin(kt) - Bk^2\cos(kt) = -k^2(A\sin(kt) + B\cos(kt)) = \frac{-25}{4} \cdot y(t).$$

Multiplying both sides by 4 yields $4y'' = -25y$ which is the desired differential equation.

3. (a) The polynomial $y^4 - 6y^3 + 5y^2$ factors as $y^2(y-1)(y-5)$. A solution of the differential equation is constant if and only if its derivative $y' = dy/dt$ is zero. In our case, we must have

$$0 = \frac{dy}{dt} = y^4 - 6y^3 + 5y^2 = y^2(y-1)(y-5)$$

and we see that the constant solutions are $y = 0$, $y = 1$ and $y = 5$.

(b) y is increasing if and only if $y' = dy/dt \geq 0$, i.e. $y^2(y-1)(y-5) \geq 0$. Since $y^2 \geq 0$ for all y, we're interested in finding when the quadratic polynomial $(y-1)(y-5)$ is nonnegative. This holds whenever y is not situated between the roots of the polynomial, i.e. for $y \leq 1$ and for $y \geq 5$.

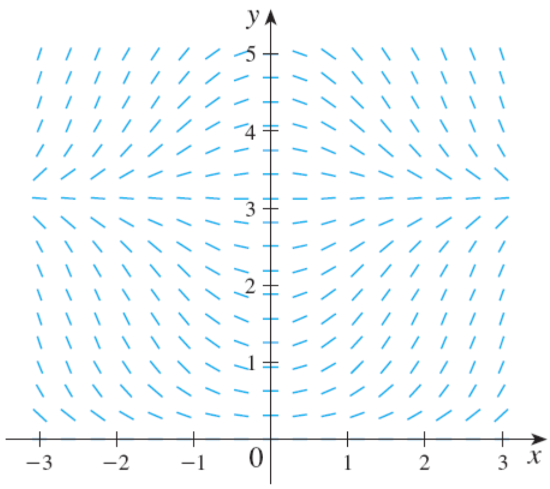
y is decreasing if and only if $y' = dy/dt \leq 0$, and the argument in the previous paragraph shows that this is possible only when $1 \leq y \leq 5$.

4. The solutions to (a) and (b) have nonnegative derivative when x, y lie in the first quadrant. But y is not always increasing for $x, y > 0$ (actually it increases for a while, and then becomes decreasing), so y' has to take on negative values. The only possibility is therefore (c).

5. A direction field for the differential equation $y' = x \sin\ y$ is shown.

   (i) Sketch the graphs of the solutions that satisfy the given initial conditions.

   (a) $y(0) = 1$. (b) $y(0) = 2$. (c) $y(0) = \pi$. (d) $y(0) = 4$. (e) $y(0) = 5$.



   (ii) Equilibrium solutions, or constant solutions, correspond to $y' = 0$. This is equivalent to $\sin(y) = 0$, which is the same as $y = n\pi$ for some integer number n.

6. Let $F(x, y) = 1 - xy$, and $h = 0.2$ be the step size. At $(x_0, y_0) = (0, 0)$, the slope is $F(x_0, y_0) = F(0, 0) = 1$. We now move on the line of slope 1 passing through $(0, 0)$ $h = 0.2$ units in the x direction. We get

$$x_1 = x_0 + h = 0.2$$

and

$$y_1 = y_0 + h \cdot F(x_0, y_0) = y_0 + h = 0.2.$$

Now we check the direction field at the new point $(x_1, y_1) = (0.2, 0.2)$. It indicates that we have to move along the line with slope $F(x_1, y_1) = 1 - 0.2 \cdot 0.2 = 0.96$ passing through (x_1, y_1). We get

$$x_2 = x_1 + h = 0.4$$

and

$$y_2 = y_1 + h \cdot F(x_1, y_1) = 0.2 + 0.2 \cdot 0.96 = 0.392.$$

Continuing in the same fashion we get

$$F(x_2, y_2) = 0.9216,\ x_3 = x_2 + h = 0.6,\ y_3 = y_2 + h \cdot F(x_2, y_2) = 0.56064$$

$$F(x_3, y_3) = 0.336384,\ x_4 = x_3 + h = 0.8,\ y_4 = y_3 + h \cdot F(x_3, y_3) = 0.6933632$$

$$F(x_4, y_4) = 0.55469056,\ x_5 = x_4 + h = 1,\ y_5 = y_4 + h \cdot F(x_4, y_4) = 0.782425088$$

Therefore, Euler's method gives us the approximate value $y(1) \approx 0.782425088$.

7. Separating the variables and integrating we obtain

$$\int e^y dy = \int \sqrt{x} dx.$$

We get

$$e^y = \frac{x^{3/2}}{3/2} + C = \frac{2x^{3/2}}{3} + C$$

or equivalently

$$y = \ln\left(\frac{2x^{3/2}}{3} + C\right).$$

8. Notice first that the constant solution $y = 0$ satisfies the differential equation. In what follows, we assume that $y \neq 0$. Separating the variables and integrating we obtain

$$\int \frac{dy}{y^2} = \int \sin x dx.$$

We get

$$\frac{-1}{y} = -\cos(x) + C$$

which is equivalent to

$$y = \frac{1}{\cos(x) - C}.$$

Notice that if we're looking for solutions defined for all values of x, then $-\cos(x) + C$ is not allowed to be 0, i.e. we must take C outside the interval $[-1, 1]$: $C > 1$ or $C < -1$.

9. Separating the variables and integrating we obtain

$$\int \frac{y^2+1}{y} dy = \int \cos x dx.$$

Since

$$\int \frac{y^2+1}{y} dy = \int y dy + \int \frac{dy}{y} = \frac{y^2}{2} + \ln|y| + K,$$

we get

$$\frac{y^2}{2} + \ln|y| = \sin(x) + C$$

for some constant C. To determine C, we use the initial condition $y(0) = 1$. We obtain

$$\frac{1}{2} + \ln(1) = \sin(0) + C, \text{ which yields } C = \frac{1}{2}.$$

It follows that y satisfies the equation

$$\frac{y^2}{2} + \ln|y| = \sin(x) + \frac{1}{2},$$

and in this case it is impossible to find a nice formula for y.

10. Separating the variables and integrating we obtain

$$\int x\cos(x)dx = \int (2y + e^{3y})dy.$$

Since

$$\int (2y + e^{3y})dy = \int 2ydy + \int e^{3y}dy = y^2 + \frac{e^{3y}}{3} + K_1,$$

and

$$\int x\cos(x)dx = x\sin(x) + \cos(x) + K_2.$$

(for calculating the last integral, use integration by parts with $u = x$ and $dv = \cos(x)dx$). We get

$$x\sin(x) + cos(x) = y^2 + \frac{e^{3y}}{3} + C$$

for some constant C. To determine C, we use the initial condition $y(0) = 0$. We obtain

$$0\sin(0) + \cos(0) = 0^2 + \frac{e^{3\cdot 0}}{3} + C = \frac{1}{3} + C, \text{ which yields } C = 1 - \frac{1}{3} = \frac{2}{3}.$$

It follows that y satisfies the equation

$$x\sin(x) + cos(x) = y^2 + \frac{e^{3y}}{3} + \frac{2}{3}$$

and again it is impossible to find a nice formula for y.

11. Differentiating $y^2 = kx^3$ we get

$$2ydy = 3kx^2dx \Leftrightarrow \frac{dy}{dx} = k\frac{3x^2}{2y},$$

and since $k = y^2/x^3$ (from the original equation), we get

$$\frac{dy}{dx} = \frac{y^2}{x^3} \cdot \frac{3x^2}{2y} = \frac{3y}{2x}.$$

The orthogonal trajectories then satisfy the differential equation

$$\frac{dy}{dx} = \frac{-1}{\frac{3y}{2x}} = \frac{-2x}{3y},$$

which is separable. We get

$$3ydy = -2xdx,$$

and by integration

$$\int 3ydy = \int -2xdx \Leftrightarrow \frac{3}{2}y^2 = -x^2 + C \Leftrightarrow y = \pm\sqrt{\frac{2}{3}(-x^2 + C)}.$$

12. Differentiating $y = \dfrac{x}{1 + kx}$ we get

$$\frac{dy}{dx} = \frac{(1 + kx) - x \cdot k}{(1 + kx)^2} = \frac{1}{(1 + kx)^2} = \frac{y^2}{x^2},$$

The orthogonal trajectories then satisfy the differential equation

$$\frac{dy}{dx} = \frac{-x^2}{y^2},$$

which is separable. We get

$$y^2 dy = -x^2 dx,$$

and by integration

$$\int y^2 dy = \int -x^2 dx \Leftrightarrow \frac{1}{3}y^3 = -\frac{1}{3}x^3 + C \Leftrightarrow y = \pm\sqrt[3]{-x^3 + C/3}.$$

13. Let $A(t)$ denote the amount of alcohol at time t, and let $c(t)$ denote the concentration of alcohol. Since the total mass of the mixture is constantly equal to 500, we get

$$c(t) = \frac{A(t)}{500}, \text{ or equivalently } A(t) = 500 \cdot c(t).$$

Since $c(0) = 4/100$, it follows that $A(0) = 500 \cdot (4/100) = 20$. Now to see how the amount of alcohol changes with time, we note that during 1 minute $5 \cdot c(t)$ gallons of alcohol exit the vat, and $5 \cdot (6/100)$ gallons enter the vat. If follows that

$$A'(t) = \frac{30}{100} - 5c(t) = \frac{30}{100} - 5 \cdot \frac{A(t)}{100} = \frac{30 - A(t)}{100}.$$

It follows that A satisfies the differential equation

$$\frac{dA}{dt} = \frac{30 - A}{100}$$

which is a separable equation. Separating variables and integrating we obtain

$$\int \frac{dA}{30 - A} = \int \frac{dt}{100}$$

or equivalently

$$-\ln|30 - A| = \frac{t}{100} + C$$

To determine C, we plug in $t = 0$, and since $A(0) = 20$ we obtain $C = -\ln(10)$. Since $30 - A$ cannot be 0 (in order for the above formula to make sense), $30 - A$ must have constant sign. But since $30 - A(0) = 10$, $30 - A$ must always be positive. We obtain $|30 - A| = 30 - A$ and therefore

$$-\ln(30 - A) = \frac{t}{100} - \ln(10)$$

or equivalently

$$\ln(30 - A) = \frac{-t}{100} + \ln(10).$$

Exponentiating, we obtain

$$30 - A = e^{\frac{-t}{100} + \ln(10)} = e^{-t/100} \cdot e^{\ln(10)}$$

which yields

$$A = 30 - 10e^{-t/100}.$$

The amount of alcohol after one hour is therefore $A(60)$, and the concentration

$$c(60) = \frac{A(60)}{500} = \frac{30 - 10e^{-60/100}}{500} = \frac{3}{50} - \frac{e^{-3/5}}{50}$$

Notice that as t approaches infinity, the concentration of alcohol approaches 6% (which shouldn't be too surprising).

14. Let $A(t)$ denote the amount of carbon dioxide at time t, and let $c(t)$ denote its concentration. Since the total amount of air is constantly equal to 180, we get

$$c(t) = \frac{A(t)}{180}, \text{ or equivalently } A(t) = 180 \cdot c(t).$$

Since $c(0) = 0.15/100 = 0.0015$, it follows that $A(0) = 180 \cdot 0.0015 = 0.27$. Now to see how the amount of carbon dioxide changes with time, we note that during 1 minute $2 \cdot c(t)$ cube meters of carbon dioxide flow out of the room, and $2 \cdot (0.05/100)$ flow into the room. If follows that

$$A'(t) = 2 \cdot \frac{0.05}{100} - 2c(t) = \frac{1}{1000} - 2 \cdot \frac{A(t)}{180} = \frac{9 - 100 \cdot A(t)}{9000}.$$

It follows that A satisfies the differential equation

$$\frac{dA}{dt} = \frac{9 - 100 \cdot A}{9000}$$

which is a separable equation. Separating variables and integrating we obtain

$$\int \frac{dA}{9 - 100A} = \int \frac{dt}{9000}$$

or equivalently

$$-\frac{\ln|9 - 100A|}{100} = \frac{t}{9000} + C$$

To determine C, we plug in $t = 0$, and since $A(0) = 0.27$ we obtain $C = -\ln|9-27|/100 = -\ln(18)/100$. Since $9-100A$ cannot be 0 (in order for the above formula to make sense), $9 - 100A$ must have constant sign. But since $9 - 100A(0) = -18$, $9 - 100A$ must always be nagative. We obtain $|9 - 100A| = 100A - 9$ and therefore

$$-\frac{\ln(100A - 9)}{100} = \frac{t}{9000} - \frac{\ln(18)}{100}$$

or equivalently

$$\ln(100A - 9) = \frac{-t}{90} + \ln(18).$$

Exponentiating, we obtain

$$100A - 9 = e^{\frac{-t}{90} + \ln(18)} = e^{-t/90} \cdot e^{\ln(18)}$$

which yields

$$100A = 9 + 18e^{-t/90} \text{ or equivalently } A = \frac{9}{100} + \frac{9e^{-t/90}}{50}.$$

The concentration of carbon dioxide is therefore

$$c(t) = \frac{A(t)}{180} = \frac{1}{2000} + \frac{e^{-t/90}}{1000}$$

As t approaches infinity $c(t)$ approaches $1/2000 = 0.05\%$.