Math 1B, Spring '10
Quiz 10, April 21

1. (3 points) Solve the initial-value problem

$$(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0, \;\; y(0) = 1.$$

First solution. The differential equation is equivalent to

$$\frac{dy}{dx} = \frac{3x(1-y)}{x^2+1}.$$

This shows that $y' = 0$ whenever $y = 1$, that is the direction field is horizontal along the line $y = 1$. It follows that any solution $y(x)$ of the differential equation which takes the value 1 at some point x_0 has to be constantly equal to 1. Since the initial condition of our problem is $y(0) = 1$, the solution must be $y \equiv 1$. $\square$

Second solution. The differential equation is equivalent to

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{3x}{x^2+1},$$

which is a linear equation, with $P(x) = Q(x) = \dfrac{3x}{x^2+1}$. We have

$$\int P(x)dx = \frac{3}{2}\int \frac{2x}{x^2+1}dx = \frac{3}{2}\ln(x^2+1).$$

It follows that the integrating factor $I(x)$ is then

$$I(x) = e^{\int P(x)dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}.$$

The general technique for solving linear differential equations yields

$$yI(x) = \int Q(x)I(x)dx = \int \frac{3x}{x^2+1}(x^2+1)^{3/2}dx = \int 3x(x^2+1)^{1/2} = (x^2+1)^{3/2} + c.$$

Dividing by $I(x)$ we get

$$y = 1 + \frac{c}{(x^2+1)^{3/2}}.$$

The initial condition $y(0) = 1$ yields $1 = 1 + c$, i.e. $c = 0$. Therefore $y \equiv 1$. $\square$

Third solution. The differential equation is separable. We get by separating the variables

$$\frac{dy}{y-1} = \frac{-3x}{x^2+1}.$$

Integrating, we obtain

$$\ln|y-1| = \frac{-3}{2}\ln(x^2+1) + c,$$

which yields by exponentiation

$$|y-1| = \frac{e^c}{(x^2+1)^{3/2}}.$$

Substituting e^c by a constant A, and allowing A to also be negative or 0, we get

$$y-1 = \frac{A}{(x^2+1)^{3/2}}.$$

The initial condition $y(0) = 1$ implies that $1-1 = A/1 = A$, hence $A = 0$ and $y \equiv 1$. $\square$

2. (4 points) Solve the initial value problem

$$2y'' + 5y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = -4.$$

Proof. The auxiliary equation

$$2r^2 + 5r + 3 = 0$$

has roots $r_1 = -1$ and $r_2 = -3/2$. It follows that the general solution of the differential equation is given by

$$y(x) = c_1 e^{-x} + c_2 e^{-3x/2}.$$

To determine the values of c_1, c_2, we use the initial conditions. Plugging in $x = 0$ in the above equation, we get

$$y(0) = 3 = c_1 + c_2.$$

We have $y'(x) = -c_1 e^{-x} - (3/2)c_2 e^{-3x/2}$, so letting $x = 0$ we obtain

$$y'(0) = -4 = -c_1 - 3c_2/2.$$

We get $c_1 = 3 - c_2$ from the first equation, and substituting in the second one we obtain

$$-4 = -3 + c_2 - 3c_2/2 \quad \Leftrightarrow \quad c_2/2 = 1 \quad \Leftrightarrow \quad c_2 = 2$$

and $c_1 = 3 - 2 = 1$. It follows that the solution to our initial value problem is

$$y(x) = e^{-x} + 2e^{-3x/2}.$$

$\square$

3. (3 points) Solve the boundary-value problem

$$y'' - 6y' + 9y = 0, \quad y(0) = 1, \quad y(1) = 0.$$

Proof. The auxiliary equation $r^2 - 6r + 9$ has $r = 3$ as a double root. Thus the general solution of the differential equation has the form

$$y(x) = (c_1 + c_2 x)e^{3x}.$$

From the boundary conditions we obtain $y(0) = 1 = c_1$ and $y(1) = 0 = (c_1 + c_2)e^3$. We get $c_2 = -c_1 = -1$ and the solution of the boundary-value problem is

$$y(x) = (1-x)e^{3x}.$$

$\square$