Math 1B, Spring '10
Quiz 11, April 28

1. (5 points) Use the method of undetermined coefficients to solve the boundary value problem

$$y'' - y' = 1 + e^x, \ y(0) = y(1) = 0.$$

Solution. The complementary equation is $y'' - y' = 0$. Its auxiliary equation is $r^2 - r = 0$, with roots $r_1 = 0$, $r_2 = 1$. It follows that the general solution of the complementary equation has the form

$$y_c = c_1 e^{0 \cdot x} + c_2 e^{1 \cdot x} = c_1 + c_2 e^x.$$

We look for a particular solution y_p of the form $y_{p_1} + y_{p_2}$, where y_{p_1} is a particular solution of the equation

$$y'' - y' = 1.$$

and y_{p_2} is a particular solution of the equation

$$y'' - y' = e^x.$$

To find y_{p_1}, notice that 1 is a constant, so the general method of undetermined coefficients says that we should look for solutions of the form $y_{p_1} = a$ with a a constant. But a is also a solution of the complementary equation, so we need to multiply it by x, i.e. take $y_{p_1} = ax$. We get

$$y'_{p_1} = a, \ y''_{p_1} = 0,$$

hence $y''_{p_1} - y'_{p_1} = -a = 1$, or equivalently $a = -1$, yielding $y_{p_1} = -x$.

Similarly, we observe that we cannot choose y_{p_2} of the form be^x because it would be a solution of the complementary equation, so we must take $y_{p_2} = bxe^x$. We get

$$y'_{p_2} = b(e^x + xe^x) = b(x+1)e^x,$$

$$y''_{p_2} = b(e^x + (x+1)e^x) = b(x+2)e^x,$$

and therefore $y''_{p_2} - y_{p_1} = be^x = e^x$. It follows that $b = 1$ and $y_{p_2} = xe^x$. The general solution of the given differential equation is therefore of the form

$$y = y_c + y_p = y_c + y_{p_1} + y_{p_2} = c_1 + c_2 e^x - x + xe^x.$$

The boundary conditions $y(0) = y(1) = 0$ yield

$$c_1 + c_2 = 0 \text{ and } c_1 + c_2 e - 1 + e = 0,$$

or equivalently $c_2 = -c_1$ and $c_1(1 - e) = 1 - e$. We get $c_1 = 1$, $c_2 = -1$ and

$$y(x) = 1 - e^x - x + xe^x = (1 - x)(1 - e^x).$$

$\square$

2. (5 points) Solve the differential equation using the method of variation of parameters.

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

Solution. The complementary equation is $y'' - 2y' + y = 0$. Its auxiliary equation is $r^2 - 2r + 1 = 0$, with roots $r_1 = r_2 = 1$. It follows that the general solution of the complementary equation has the form

$$y_c = c_1 y_1 + c_2 y_2,$$

where $y_1 = e^x$, $y_2 = xe^x$. The general solution of the given differential equation can be written as

$$y = y_c + y_p$$

where y_p is a particular solution. Using the method of variation of parameters, we search for y_p of the form $u_1 y_1 + u_2 y_2$, with u_1, u_2 functions whose derivatives satisfy the system of equations

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = F(x) = \dfrac{e^x}{1+x^2} \end{cases}$$

Substituting y_1, y_2 in the above system we obtain

$$\begin{cases} u_1' e^x + u_2' x e^x = 0 \\ u_1' e^x + u_2'(x+1)e^x = \dfrac{e^x}{1+x^2} \end{cases}$$

Subtracting the first equation from the second, we get

$$u_2' e^x = \frac{e^x}{1+x^2} \Rightarrow u_2' = \frac{1}{1+x^2} \Rightarrow u_2 = \arctan(x).$$

Dividing by e^x in the first equation we get

$$u_1' = -u_2' x = -\frac{x}{1+x^2} \Rightarrow u_1 = \frac{-1}{2}\ln(1+x^2).$$

It follows that

$$y_p = \frac{-1}{2}\ln(1+x^2)e^x + \arctan(x)xe^x$$

and

$$y = y_c + y_p = e^x\left(c_1 + c_2 x - \frac{1}{2}\ln(1+x^2) + x\arctan(x)\right).$$

□

3. (1 point) Use power series to solve the initial value problem

$$y'' - xy' = x, \ y(0) = 0, \ y'(0) = 0.$$

(1 point) Give a closed formula for y' when y is the solution of the above problem.

Solution. We look for a solution $y = \sum_{n=0}^{\infty} a_n x^n$. The initial conditions $y(0) = y'(0) = 0$ yield $a_0 = a_1 = 0$. We have

$$y' = \sum_{n\geq 0} n a_n x^{n-1}$$

hence

$$xy' = \sum_{n\geq 0} n a_n x^n.$$

Also,

$$y'' = \sum_{n\geq 0} n(n-1)a_n x^{n-2} = \sum_{n\geq 2} n(n-1)a_n x^{n-2} = \sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n.$$

We get

$$y'' - xy' = \sum_{n\geq 0}((n+2)(n+1)a_{n+2} - n a_n)x^n = x.$$

The above equality is equivalent to

$$(n+2)(n+1)a_{n+2} = n a_n \text{ for } n \neq 1, \text{ and } 6a_3 - a_1 = 1.$$

We obtain the recursion relation

$$a_{n+2} = \frac{n}{(n+2)(n+1)} a_n, \; n \geq 2,$$

and $a_2 = 0$, $a_3 = 1/6$. It follows that $a_{2n} = 0$ and for $n \geq 1$

$$\begin{aligned}
a_{2n+1} &= \frac{2n-1}{(2n+1)(2n)} a_{2n-1} = \frac{2n-1}{(2n+1)(2n)} \cdot \frac{2n-3}{(2n-1)(2n-2)} a_{2n-3} = \cdots \\
&= \frac{2n-1}{(2n+1)(2n)} \cdot \frac{2n-3}{(2n-1)(2n-2)} \cdots a_3 \\
&= \frac{(2n-1)(2n-3)\cdots 3}{(2n+1)(2n)(2n-1)(2n-2)\cdots 5\cdot 4} \cdot \frac{1}{6} \\
&= \frac{1}{2n+1} \cdot \frac{1}{2n(2n-2)(2n-4)\cdots 4\cdot 2} \\
&= \frac{1}{2n+1} \cdot \frac{1}{2^n(n!)}
\end{aligned}$$

We get

$$y = \sum_{n\geq 1} \frac{1}{2^n n!} \cdot \frac{x^{2n+1}}{2n+1}$$

and

$$y' = \sum_{n\geq 1} \frac{x^{2n}}{2^n n!} = \sum_{n\geq 1} \frac{(x^2/2)^n}{n!} = e^{x^2/2} - 1.$$

To get y' you can also make the substitution $z = y'$, and obtain that z is the solution of the linear 1st order differential equation

$$z' - xz = x,$$

which you can solve using the methods of chapter 9. $\square$