Math 1B, Spring '10
Quiz 2, February 3

1. (4 points) Use your favorite method to evaluate

$$\int_0^1 \sqrt{1-x^2}dx.$$

First solution. Letting $u = \sqrt{1-x^2}$, $dv = dx$, we have $du = \dfrac{-x}{\sqrt{1-x^2}}$ and $v = x$. Using integration by parts we get

$$\int_0^1 \sqrt{1-x^2}dx = x\sqrt{1-x^2}\,|_0^1 - \int_0^1 \frac{-x^2}{\sqrt{1-x^2}}dx.$$

The function $x\sqrt{1-x^2}$ is zero when $x = 0, 1$, so we are left with evaluating

$$\begin{aligned}
-\int_0^1 \frac{-x^2}{\sqrt{1-x^2}}dx &= -\int_0^1 \frac{1-x^2}{\sqrt{1-x^2}}dx + \int_0^1 \frac{1}{\sqrt{1-x^2}}dx \\
&= -\int_0^1 \sqrt{1-x^2}dx + \arcsin(x)\,|_0^1 \\
&= -\int_0^1 \sqrt{1-x^2}dx + \frac{\pi}{2}.
\end{aligned}$$

It follows that $2\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{2}$, or $\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$.

$\square$

Second solution. The substitution $x = \sin(y)$ yields $\sqrt{1-x^2} = \cos(y)$ and $dx = \cos(y)dy$. Therefore

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}dx &= \int_0^{\frac{\pi}{2}} \cos^2(y)dy \\
&= \int_0^{\frac{\pi}{2}} \frac{1+\cos(2y)}{2}dy \\
&= \left(\frac{y}{2} + \frac{\sin(2y)}{4}\right)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{4}.
\end{aligned}$$

$\square$

Third solution. The arc of the unit circle contained in the first quadrant can be parameterized by $x \mapsto \sqrt{1-x^2}$, $x \in [0,1]$. It follows that the value of $\int_0^1 \sqrt{1-x^2}dx$ is precisely the area of the portion of the unit disk which is contained in the first quadrant. This is equal to one-fourth of the total area of the unit disk, i.e. $\frac{\pi}{4}$. $\square$

2. (3 points) Evaluate

$$\int \frac{10}{(x-1)(x^2+9)}dx$$

Solution. We look for constants a, b, c such that

$$\frac{10}{(x-1)(x^2+9)} = \frac{a}{x-1} + \frac{bx+c}{x^2+9}$$

Multiplying both sides by $(x-1)(x^2+9)$ we get that the above equality is equivalent to
$10 = a(x^2+9)+(bx+c)(x-1) = ax^2+9a+bx^2-bx+cx-c = (a+b)x^2+(c-b)x+(9a-c)$

We must then have $a+b=0$, $c-b=0$ and $9a-c=10$. It follows that $c=b=-a$ and, replacing c by $-a$ in the last equality, that $9a+a=10$. We get $a=1$, $c=b=-1$ and therefore

$$\begin{aligned}\int \frac{10}{(x-1)(x^2+9)}dx &= \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+9}dx \\ &= \ln|x-1| - \int \frac{x}{x^2+9}dx - \int \frac{1}{x^2+9}dx \\ &= \ln|x-1| - \frac{\ln|x^2+9|}{2} - \frac{1}{3}\arctan(x/3) + C\end{aligned}$$

□

3. (3 points) Make a substitution to express the integrand as a rational function and then evaluate the integral

$$\int \frac{\sec^2 t}{\tan^2 t + 3\tan\ t + 2}dt$$

Proof. Let $u = \tan\ t$. We get $du = \sec^2 t dt$ and therefore

$$\int \frac{\sec^2 t}{\tan^2 t + 3\tan\ t + 2}dt = \int \frac{du}{u^2+3u+2}$$

To integrate the rational function $1/(u^2+3u+2)$ we first factor the denominator as $(u+1)(u+2)$. Then we look for constants a, b such that

$$\frac{1}{u^2+3u+2} = \frac{a}{u+1} + \frac{b}{u+2}$$

Multiplying both sides by $u^2+3u+2 = (u+1)(u+2)$ we get that the above equality is equivalent to

$$1 = a(u+2) + b(u+1) \tag{1}$$

Plugging in $u=-1$ we get $1 = a(-1+2) = a$. Plugging in $u=-2$ we get $1 = b(-2+1) = -b$, so $b=-1$. Therefore

$$\int \frac{du}{u^2+3u+2} = \int \frac{du}{u+1} - \int \frac{du}{u+2} = \ln|u+1| - \ln|u+2| + C = \ln\left|\frac{u+1}{u+2}\right| + C$$

Going back to the variable t, we obtain

$$\int \frac{\sec^2 t}{\tan^2 t + 3\tan\ t + 2}dt = \ln\left|\frac{\tan\ t+1}{\tan\ t+2}\right| + C$$

□