Math 1B, Spring '10
Quiz 3, February 10

1. (3 points) Evaluate

$$\int \theta \tan^2 \theta d\theta$$

*Solution.* Since $\theta$ is an Algebraic function and $\tan^2\theta$ is Trigonometric, LIATE tells us that we have to use integration by parts with

$$u = \theta, \; dv = \tan^2 \theta d\theta.$$

Then $du = d\theta$ and

$$v = \int \tan^2 \theta d\theta = \int (\sec^2\theta - 1) d\theta = \tan\theta - \theta$$

We get

$$\begin{aligned} \int \theta \tan^2 \theta d\theta = uv - \int v du &= \theta(\tan\theta - \theta) - \int (\tan\theta - \theta) d\theta \\ &= \theta \tan\theta - \theta^2 + \ln|\cos\theta| + \frac{\theta^2}{2} + C \\ &= \theta\tan\theta - \frac{\theta^2}{2} + \ln|\cos\theta| + C \end{aligned}$$

□

2. (4 points) Use the Trapezoid Rule and Simpson's Rule to approximate

$$\int_0^\pi \sin^2 x dx$$

using $n = 4$ intervals.

*Solution.* The endpoints of the interval are $a = 0$ and $b = \pi$, $\Delta x = \frac{b-a}{4} = \pi/4$ so the intermediate points $x_0, x_1, x_2, x_3, x_4$ are then equal to $0, \pi/4, \pi/2, 3\pi/4, \pi$.
Using the Trapezoid Rule, we get

$$\int_0^\pi \sin^2 x dx \approx T_4 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$

Using that $\sin(0) = \sin(\pi) = 0$, $\sin(\pi/2) = 1$ and $\sin(\pi/4) = \sin(3\pi/4) = 1/\sqrt{2}$ we get $f(x_0) = f(x_4) = 0$, $f(x_2) = 1$ and $f(x_1) = f(x_3) = 1/2$. Putting everything together we have

$$T_4 = \frac{\pi/4}{2}(0 + 1 + 2 + 1 + 0) = \frac{\pi}{2}$$

Using now Simpson's Rule, we have

$$\int_0^\pi \sin^2 x dx \approx S_4 = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) = \frac{\pi/4}{3}(0+2+2+2+0) = \frac{\pi}{2}$$

Notice that the actual value of $\int_0^\pi \sin^2 x dx$ is also $\pi/2$. Isn't that weird? □

3. (3 points) Determine whether the following integral is convergent or divergent. If convergent, evaluate it.

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx$$

*Solution.* The substitution $u = e^x$ yields $du = e^x dx$ and therefore

$$\int \frac{e^x}{e^{2x}+3}dx = \int \frac{du}{u^2+3} = \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right) + C = \frac{\arctan(e^x/\sqrt{3})}{\sqrt{3}} + C$$

We get

$$\begin{aligned}\int_0^\infty \frac{e^x}{e^{2x}+3}dx &= \lim_{t\to\infty}\int_0^t \frac{e^x}{e^{2x}+3}dx = \lim_{t\to\infty}\left[\frac{\arctan(e^x/\sqrt{3})}{\sqrt{3}}\right]_0^t \\ &= \lim_{t\to\infty}\left(\frac{\arctan(e^t/\sqrt{3})}{\sqrt{3}} - \frac{\arctan(1/\sqrt{3})}{\sqrt{3}}\right)\end{aligned}$$

Since $e^t/\sqrt{3} \to \infty$ as $t \to \infty$ we get that $\arctan(e^t/\sqrt{3}) \to \pi/2$. Using the fact that $\arctan(1/\sqrt{3}) = \pi/6$ we get

$$\lim_{t\to\infty}\left(\frac{\arctan(e^t/\sqrt{3})}{\sqrt{3}} - \frac{\arctan(1/\sqrt{3})}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}} - \frac{\pi}{6\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$$

so

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx = \frac{\pi}{3\sqrt{3}}$$

is convergent. $\square$