Math 1B, Spring '10
Quiz 4, February 17

1. (3 points) Find the length of the curve

$$y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x}),\ 0 \le x \le 1$$

*Solution.* The length of the curve is given by the arc length formula:

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{x-x^2}} = \frac{2(1-x)}{2\sqrt{x-x^2}} \\ &= \sqrt{\frac{(1-x)^2}{x-x^2}} = \sqrt{\frac{1-x}{x}}, \end{aligned}$$

hence

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1-x}{x} = \frac{1}{x}$$

We get

$$L = \int_0^1 \sqrt{\frac{1}{x}} dx = [2\sqrt{x}]_0^1 = 2 - 0 = 2$$

$\square$

2. (4 points) Find the area of the surface obtained by rotating the curve

$$x = \frac{1}{3}(y^2+2)^{3/2},\ 1 \le y \le 2$$

about the $x$-axis.

*Solution.* Since we're rotating about the $x$-axis, the formula for the area is given by

$$A = 2\pi \int y\sqrt{(dx)^2 + (dy)^2}$$

Since we're given $x$ in terms of $y$, we'll write the integral in terms of the variable $y$. Notice first that

$$\frac{dx}{dy} = \frac{1}{3} \cdot \frac{3}{2}(y^2+2)^{3/2-1} \cdot (2y) = y\sqrt{y^2+2},$$

therefore

$$
\begin{aligned}
A &= 2\pi \int_1^2 y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}dy \\
&= 2\pi \int_1^2 y\sqrt{(y\sqrt{y^2+2})^2 + 1}dy = 2\pi \int_1^2 y\sqrt{y^4 + 2y^2 + 1}dy \\
&= 2\pi \int_1^2 y\sqrt{(y^2+1)^2}dy = 2\pi \int_1^2 (y^3 + y)dy = 2\pi \left[\frac{y^4}{4} + \frac{y^2}{2}\right]_1^2 \\
&= 2\pi \left(\frac{2^4}{4} + \frac{2^2}{2} - \frac{1^4}{4} - \frac{1^2}{2}\right) = 2\pi \frac{21}{4} = \frac{21\pi}{2}
\end{aligned}
$$

□

3. (3 points) Find the centroid of the region bounded by the curves

$$y = x^2, x = y^2$$

*Solution.* The region is bounded by the graphs of the functions $g(x) = x^2 \leq f(x) = \sqrt{x}$, $0 \leq x \leq 1$. The area of this region is

$$A = \int_0^1 (f(x) - g(x))dx = \int_0^1 \sqrt{x}dx - \int_0^1 x^2 dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

The coordinates of the center of mass are then given by

$$\overline{x} = \frac{1}{A}\int_0^1 x(f(x) - g(x))dx = 3\int_0^1 x(\sqrt{x} - x^2)dx = 3\left[\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^1 = 3\left(\frac{2}{5} - \frac{1}{4}\right) = \frac{9}{20}$$

Now since the region is symmetric with respect to the $x = y$ line, the centroid has to be contained in this line, hence $\overline{y} = \overline{x} = \frac{9}{20}$. Alternatively

$$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(f(x)^2 - g(x)^2)dx = \frac{3}{2}\int_0^1 (x - x^4)dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{9}{20}$$

□