Math 1B, Spring '10
Quiz 5, March 3

1. (3 points) Is the sequence

$$a_n = \ln(2n^2+1) - \ln(n^2+1)$$

increasing, decreasing or not monotonic? Determine whether it's convergent, and if so find its limit.

Solution. Let $f(x) = \ln(2x^2+1) - \ln(x^2+1)$. We have

$$f'(x) = \frac{4x}{2x^2+1} - \frac{2x}{x^2+1} = \frac{4x(x^2+1) - 2x(2x^2+1)}{(2x^2+1)(x^2+1)} = \frac{2x}{(2x^2+1)(x^2+1)} \geq 0 \text{ for } x \geq 0$$

It follows that the function $f(x)$ is increasing for $x \geq 0$, hence the sequence $a_n = f(n)$ is also increasing.

To determine whether the sequence is convergent, notice that

$$a_n = \ln\left(\frac{2n^2+1}{n^2+1}\right).$$

Since

$$\lim_{n\to\infty} \frac{2n^2+1}{n^2+1} = 2$$

we get that a_n is convergent and

$$\lim_{n\to\infty} a_n = \ln\left(\lim_{n\to\infty} \frac{2n^2+1}{n^2+1}\right) = \ln(2)$$

□

2. (4 points) Determine whether the series

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} + \frac{2}{n^2+2n}\right)$$

is convergent or divergent. If it is convergent, find its sum.

Solution. Let $a_n = 3/2^n$ and $b_n = 2/(n^2+2n)$. Observe that

$$\sum_{n=1}^{\infty} a_n = \frac{3}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \cdots$$

is a geometric series with initial term $a = 3/2$ and common ratio $r = \frac{1}{2}$. Therefore the series is convergent, and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3}{2^n} = \frac{a}{1-r} = \frac{3/2}{1-(1/2)} = 3$$

To compute $\sum b_n$, notice that

$$b_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2},$$

so $\sum b_n$ is telescoping. We get

$$\sum_{n=1}^{\infty} b_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots$$

and we see that all terms except for $\frac{1}{1}$ and $\frac{1}{2}$ cancel out. Therefore

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^2+2n} = 1 + \frac{1}{2} = \frac{3}{2}$$

It follows that the initial series can be written as a sum of two convergent series, hence is itself convergent, and

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} + \frac{2}{n^2+2n}\right) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = 3 + \frac{3}{2} = \frac{9}{2}$$

□

3. (3 points) Determine whether the series

$$\sum_{n=1}^{\infty} ne^{-n}$$

is convergent or divergent.

Solution. Let $f(x) = xe^{-x}$. f is continuous, positive and

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} \leq 0 \text{ for } x \geq 1$$

so f is increasing for $x \leq 1$. We can therefore apply the Integral Test to conclude that $\sum_{n=1}^{\infty} ne^{-n}$ is convergent if and only if the improper integral

$$\int_1^{\infty} xe^{-x} dx$$

is convergent. To calculate $\int xe^{-x}dx$, we use integration by parts, $u = x$, $dv = e^{-x}$, $du = dx$, $v = -e^{-x}$. We get

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$$

so

$$\int_1^{\infty} xe^{-x}dx = \lim_{t\to\infty} [-xe^{-x} - e^{-x}]_1^t = \lim_{t\to\infty} (-te^{-t} - e^{-t} + 2e^{-1}) = 2e^{-1}.$$

The integral is therefore convergent, hence so is the series

$$\sum_{n=1}^{\infty} ne^{-n}$$

□