Math 1B, Section 205, Spring '10
Quiz 6, March 10

1. (3 points) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$$

is convergent or divergent.

Solution. Let $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$. The rate of growth of the numerator is controlled by the term n^2, and the rate of growth of the denominator is controlled by the term n^6. We conclude that as n goes to infinity, a_n is roughly $n^2/\sqrt{n^6} = 1/n$. This suggests the use of the Limit Comparison Test, with $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n\to\infty} \frac{n^3\left(\dfrac{1}{n^2}+\dfrac{1}{n}+1\right)}{n^3\sqrt{\dfrac{1}{n^6}+\dfrac{1}{n^4}+1}} = 1$$

(the last equality follows by canceling n^3 and observing that as $n \to \infty$, $1/n^2, 1/n, 1/n^6, 1/n^4 \to 0$).

It follows by the Limit Comparison Test that the series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} b_n$ behave the same. The latter series is the Harmonic Series, which is divergent, hence $\sum_{n\geq 1} a_n$ is also divergent. $\square$

2. (4 points) Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$$

is absolutely convergent, conditionally convergent, or divergent.

Solution. Let $a_n = \dfrac{n^2}{n^3+4}$. We have $\lim_{n\to\infty} a_n = 0$ and the sequence a_n is decreasing starting with the second term, because the function $f(x) = x^2/(x^3+4)$ has negative derivative for $x \geq 2$:

$$f'(x) = \frac{2x(x^3+4) - x^2 \cdot 3x^2}{(x^3+4)^2} = \frac{-x^4+8x}{(x^3+4)^2} = \frac{x(2^3-x^3)}{(x^3+4)^2} \leq 0 \text{ for } x \geq 2$$

It follows by the Alternating Series Test that the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$$

is convergent. However, the series is not absolutely convergent. To see this, notice that a_n is roughly $n^2/n^3 = 1/n$ as n approaches infinity. Using the Limit Comparison Theorem with $b_n = 1/n$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3+4} = 1,$$

hence $\sum_{n\geq 1} a_n$ behaves like the Harmonic Series, and is therefore divergent.

We conclude that the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$$

is conditionally convergent. $\square$

3. (3 points) Determine whether the series

$$\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$$

is convergent or divergent.

Solution. Let $a_n = (\sqrt[n]{n} - 1)^n$. We test the convergence of the series using the Root Test. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (\sqrt[n]{n} - 1) = 1 - 1 = 0 < 1.$$

According to the Root Test, the series $\sum a_n = \sum (\sqrt[n]{n} - 1)^n$ is convergent.

(To prove that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ it suffices to show that

$$\ln(\lim_{n\to\infty} \sqrt[n]{n}) = 0.$$

But ln is a continuous function, hence

$$\ln(\lim_{n\to\infty} \sqrt[n]{n}) = \lim_{n\to\infty} \ln(n^{1/n}) = \lim_{n\to\infty} \frac{1}{n} \ln\ n = \lim_{x\to\infty} \frac{\ln\ x}{x} \overset{\text{l'Hôpital}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0,$$

which is what we wanted to show.) $\square$