Math 1B, Section 205, Spring '10
Quiz 7, March 17

1. (3 points) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2 \cdot 2^{n-1}}{(-5)^n}$$

is convergent or divergent.

First solution. Let $a_n = \dfrac{n^2 \cdot 2^{n-1}}{(-5)^n}$. We use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\dfrac{(n+1)^2 \cdot 2^n}{5^{n+1}}}{\dfrac{n^2 \cdot 2^{n-1}}{5^n}} = \lim_{n\to\infty} \frac{2(n+1)^2}{5n^2} = \frac{2}{5} < 1$$

It follows that the series is absolutely convergent, hence convergent. $\square$

Second solution. Let $a_n = \dfrac{n^2 \cdot 2^{n-1}}{(-5)^n}$. We use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2 \cdot 2^{n-1}}{5^n}} = \lim_{n\to\infty} (\sqrt[n]{n})^2 \cdot 2^{(n-1)/n} \cdot 5^{-n/n} = 1^2 \cdot 2^1 \cdot 5^{-1} = \frac{2}{5} < 1$$

It follows that the series is absolutely convergent, hence convergent. $\square$

2. (4 points) Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n(1-x)^n}{n^2+1}.$$

Solution. We let

$$a_n = \frac{n(1-x)^n}{n^2+1}$$

and use the Ratio Test to determine the radius of convergence. We have

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)(1-x)^{n+1}}{(n+1)^2+1}}{\dfrac{n(1-x)^n}{n^2+1}} = \frac{n+1}{n} \cdot \frac{n^2+1}{n^2+2n+2} \cdot (1-x).$$

Since $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = 1$ and $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} = 1$ we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |1-x|.$$

According to the Ratio Test, convergence of the series holds for $|1-x|<1$ (which is the same as $|x-1|<1$). It follows that the radius of convergence is equal to 1, and the interval of convergence contains the interval $(1-1,1+1)=(0,2)$. In order to compute the interval of convergence, we only need to test whether the series is convergent or divergent when $x=0,2$.

If $x=0$ we get

$$\sum_{n=1}^{\infty}\frac{n(1-0)^n}{n^2+1}=\sum_{n=1}^{\infty}\frac{n}{n^2+1}.$$

This behaves the same as $\sum\frac{n}{n^2}=\sum\frac{1}{n}$ (Limit Comparison Test - you need to provide the missing details here..what is the series you compare to, why is this series divergent, why does it behave the same as the original series?), which is divergent by the p-series test.

If $x=2$ we get

$$\sum_{n=1}^{\infty}\frac{n(1-2)^n}{n^2+1}=\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1},$$

which is convergent by the Alternating Series Test (you need to provide the missing details here..decreasing, positive).

In conclusion, the radius of convergence is $R=1$ and the interval of convergence is $I=(0,2]$.

Homework: repeat the proof using the Root Test instead of the Ratio Test (you'll have to prove that $\lim_{n\to\infty}\sqrt[n]{n^2+1}=1$). □

3. (3 points) Find the power series representation for

$$f(x)=\arctan(x/3).$$

Solution. We have

$$f'(x)=\frac{1}{1+(x/3)^2}\cdot\frac{1}{3}=\frac{1}{3}\cdot\frac{1}{1-(-x^2/9)}=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x^2}{9}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+1}}x^{2n}$$

It follows that

$$f(x)=\int f'(x)dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+1}}\cdot\frac{x^{2n+1}}{2n+1}+C.$$

To determine C, we plug in $x=0$ and get $f(0)=0+C$, i.e. $C=0$. It follows that

$$f(x)=\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{3^{2n+1}(2n+1)}\right)x^{2n+1}\quad\square$$