Math 1B, Section 201, Spring '10
Quiz 8, March 31

1. (3 points) Evaluate the indefinite integral
$$\int \frac{x}{\sqrt{x^2+4}}dx$$
as a power series.

Solution. Using the Binomial Expansion, we get
$$\begin{aligned}\frac{x}{\sqrt{x^2+4}} &= \frac{x}{2}\cdot\left(1+\frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x^2}{4}\right)^n \\ &= \frac{x}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n}}{2^{2n}} = \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n+1}}{2^{2n+1}}\end{aligned}$$
It follows that
$$\int \frac{x}{\sqrt{x^2+4}}dx = \int\left(\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n+1}}{2^{2n+1}}\right) = \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n+2}}{2^{2n+2}(n+1)} + C.$$
Remark. Using the fact that
$$\binom{-1/2}{n}\frac{1/2}{n+1} = \binom{1/2}{n+1}$$
you can rewrite the last series as
$$2\sum_{n=0}^{\infty}\binom{1/2}{n+1}\frac{x^{2n+2}}{2^{2n+2}} = 2\sum_{n=0}^{\infty}\binom{1/2}{n+1}\left(\frac{x^2}{4}\right)^{n+1} = 2((1+x^2/4)^{1/2}-1) = \sqrt{x^2+4}-2$$
which shouldn't be a surprise, since you know that $\int \frac{x}{\sqrt{x^2+4}}dx = \sqrt{x^2+4}$ up to a constant. □

2. (4 points) Use series to evaluate the limit
$$\lim_{x\to 0}\frac{\sin\ x - x + \frac{1}{6}x^3}{x^5}.$$
Solution. Using Taylor's inequality we have that
$$\lim_{x\to 0}\frac{\sin\ x - T_5(x)}{x^5} = 0,$$
therefore
$$\lim_{x\to 0}\frac{\sin\ x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0}\frac{(\sin\ x - T_5(x)) + (T_5(x) - x + \frac{1}{6}x^3)}{x^5} = \lim_{x\to 0}\frac{T_5(x) - x + \frac{1}{6}x^3}{x^5}.$$

The Taylor expansion of sin x is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

hence

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}.$$

It follows that

$$\frac{T_5(x) - x + \frac{1}{6}x^3}{x^5} = \frac{1}{120},$$

and therefore

$$\lim_{x \to 0} \frac{\sin\ x - x + \frac{1}{6}x^3}{x^5} = \frac{1}{120}.$$

□

3. (3 points) Approximate $f(x) = x\ln(x)$ by a Taylor polynomial of degree $n = 3$ at the number $a = 1$. Use Taylor's inequality to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when x lies in the interval $[0.5, 1.5]$.

Solution. We have $f(1) = 1 \cdot \ln(1) = 0$,

$$f'(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x), \text{ so } f'(1) = 1 + \ln(1) = 1$$
$$f''(x) = \frac{1}{x}, \text{ so } f''(1) = 1 \text{ and } f'''(x) = \frac{-1}{x^2}, \text{ so } f'''(1) = -1$$

It follows that

$$T_3(x) = \frac{1}{1!}(x-1) + \frac{1}{2!}(x-1)^2 - \frac{1}{3!}(x-1)^3 = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6}.$$

Taylor's inequality tells us that

$$|R_3(x)| = |f(x) - T_3(x)| \le \frac{M}{4!}|x-1|^4$$

where M is an upper bound for $f^{(4)}(x)$ on the given interval $[0.5, 1.5]$. We have

$$f^{(4)}(x) = \frac{2}{x^3}$$

which is decreasing and positive on the interval $[0.5, 1.5]$, so its maximal value is attained at the left endpoint of the interval, $x = 0.5$. We can then take

$$M = f^{(4)}(0.5) = \frac{2}{0.5^3} = 16$$

so that Taylor's inequality becomes

$$|R_3(x)| \le \frac{16}{24}|x-1|^4$$

Since $|x - 1| \le 0.5$ for $x \in [0.5, 1.5]$, we get

$$|R_3(x)| \le \frac{2}{3}|x-1|^4 \le \frac{2}{3} \cdot 0.5^4 = \frac{2}{3} \cdot \frac{1}{16} = \frac{1}{24}$$

so the approximation is accurate to within $1/24 \approx 0.0416$. □