Math 1B, Section 205, Spring '10
Quiz 8, March 31

1. (4 points) Use series to evaluate the limit

$$\lim_{x\to 0} \frac{\sin\ x - x + \frac{1}{6}x^3}{x^5}.$$

Solution. Using Taylor's inequality we have that

$$\lim_{x\to 0} \frac{\sin\ x - T_5(x)}{x^5} = 0,$$

therefore

$$\lim_{x\to 0} \frac{\sin\ x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{(\sin\ x - T_5(x)) + (T_5(x) - x + \frac{1}{6}x^3)}{x^5} = \lim_{x\to 0} \frac{T_5(x) - x + \frac{1}{6}x^3}{x^5}.$$

The Taylor expansion of $\sin\ x$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

hence

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}.$$

It follows that

$$\frac{T_5(x) - x + \frac{1}{6}x^3}{x^5} = \frac{1}{120},$$

and therefore

$$\lim_{x\to 0} \frac{\sin\ x - x + \frac{1}{6}x^3}{x^5} = \frac{1}{120}.$$

□

2. (3 points) Use multiplication of power series to find the first three nonzero terms in the Maclaurin series for the function

$$y = e^{-x^2} \cos\ x.$$

Solution. We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

hence

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \cdots .$$

The Taylor expansion for $\cos\ x$ is

$$\cos\ x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

We get that

$$y = (1 - x^2 + \frac{x^4}{2} - \cdots) \cdot (1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots)$$

To get the constant term in y we have to multiply the constant terms in each of the two factors, i.e. $1 \cdot 1 = 1$. Since all the exponents that occur in the two expansions are even, y consists only of terms of even exponent.

To get the coefficient of x^2 in the expression of y, we add the product of the constant term in the first factor and the coefficient of x^2 from the second factor to the product of the coefficient of x^2 from the first factor with the constant term in the second factor, i.e. $1 \cdot (-1/2) + (-1) \cdot 1 = -3/2$.

Similarly, the coefficient of x^4 is $1 \cdot (1/24) + (-1)(-1/2) + (1/2) \cdot 1 = 25/24$. It follows that

$$y = 1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \cdots$$

$\square$

3. (3 points) Approximate $f(x) = x\sin(x)$ by a Taylor polynomial of degree $n = 4$ at the number $a = 0$. Use Taylor's inequality to estimate the accuracy of the approximation $f(x) \approx T_4(x)$ when x lies in the interval $[-1, 1]$.

Solution. We have $f(0) = 0$,

$$\begin{aligned} f'(x) &= x\cos(x) + \sin(x), \text{ so } f'(0) = 0, \; f''(x) = -x\sin(x) + 2\cos(x), \text{ so } f''(0) = 2 \\ f'''(x) &= -(x\cos(x) + \sin(x)) - 2\sin(x), \text{ so } f'''(0) = 0 \text{ and} \\ f^{(4)}(x) &= x\sin(x) - 4\cos(x), \text{ so } f^{(4)}(0) = -4 \end{aligned}$$

It follows that

$$T_4(x) = \frac{2}{2!}x^2 - \frac{4}{4!}x^4 = x^2 - \frac{x^4}{6}.$$

Taylor's inequality tells us that

$$|R_4(x)| = |f(x) - T_4(x)| \leq \frac{M}{5!}|x|^5$$

where M is an upper bound for $f^{(5)}(x)$ on the given interval $[-1, 1]$. We have

$$|f^{(5)}(x)| = |x\cos(x) + 5\sin(x)| \leq |x| \cdot |\cos(x)| + 5|\sin(x)| \leq 1 \cdot 1 + 5 = 6$$

so we can take $M = 6$. Taylor's inequality becomes

$$|R_4(x)| \leq \frac{6}{120}|x|^5$$

Since $|x| \leq 1$ for $x \in [-1, 1]$, we get

$$|R_4(x)| \leq \frac{1}{20}|x|^5 \leq \frac{1}{20} \cdot 1^5 = 0.05$$

so the approximation is accurate to within 0.05.