Math 1B, Spring '10
Quiz 9, April 14

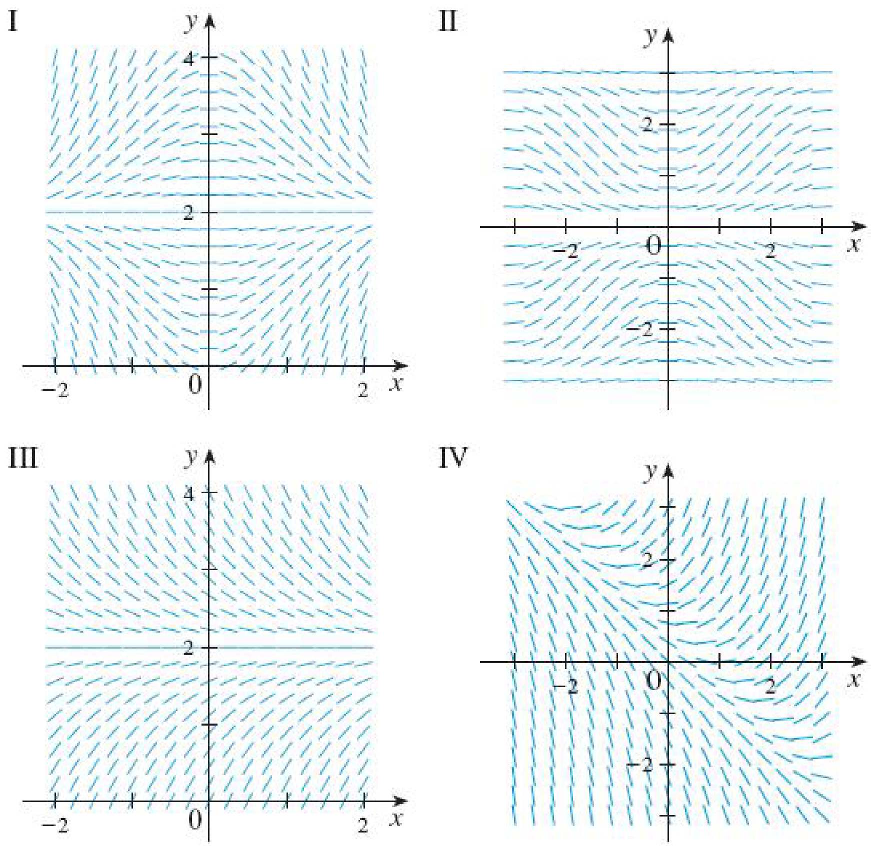
1. (3 points) Match the differential equation with its direction field. Give reasons for your answer.

   (a) $y' = 2 - y$

   (b) $y' = x(2 - y)$

   (c) $y' = x + y - 1$

   (d) $y' = \sin(x)\sin(y)$



Solution. (a) corresponds to III, because the slopes in the direction field (i.e. the values of y') are independent of x.

(b) corresponds to I, because the direction field is horizontal ($y' = 0$) whenever $x = 0$ or $y = 2$

(c) corresponds to IV because the slope at $(0, 0)$ is -1.

(d) corresponds to II, because the direction field is horizontal at $y = \pm\pi$.

□

2. (4 points) Find the solution of the differential equation

$$xy' + y = y^2$$

that satisfies the initial condition $y(1) = -1$.

Solution. Note first that the equation is separable:

$$xy' = y^2 - y \Leftrightarrow \frac{y'}{y^2 - y} = \frac{1}{x}.$$

Integrating we get

$$\int \frac{dy}{y(y-1)} = \int \frac{dx}{x},$$

or equivalently

$$\ln \left| \frac{y-1}{y} \right| = \ln|x| + c.$$

Exponentiating, we get

$$\left| \frac{y-1}{y} \right| = k|x|.$$

From the initial condition $y(1) = -1$ we obtain $k = 2$. Eliminating the absolute value, we get

$$1 - \frac{1}{y} = \pm 2x,$$

but when $x = 1$, $y = -1$, so we have to choose the " $+$ " sign. We have

$$1 - \frac{1}{y} = 2x \Leftrightarrow \frac{1}{y} = 1 - 2x \Leftrightarrow y = \frac{1}{1-2x}.$$

□

3. (3 points) Find the orthogonal trajectories of the family of curves

$$y^2 = kx^3$$

Solution. Differentiating $y^2 = kx^3$ we get

$$2ydy = 3kx^2dx \Leftrightarrow \frac{dy}{dx} = k\frac{3x^2}{2y},$$

and since $k = y^2/x^3$, we get

$$\frac{dy}{dx} = \frac{y^2}{x^3} \cdot \frac{3x^2}{2y} = \frac{3y}{2x}.$$

The orthogonal trajectories then satisfy the differential equation

$$\frac{dy}{dx} = \frac{-1}{\frac{3y}{2x}} = \frac{-2x}{3y},$$

which is separable. We get

$$3ydy = -2xdx,$$

and by integration

$$\int 3ydy = \int -2xdx \Leftrightarrow \frac{3}{2}y^2 = -x^2 + C \Leftrightarrow y = \pm\sqrt{\frac{2}{3}(-x^2 + C)}.$$

□