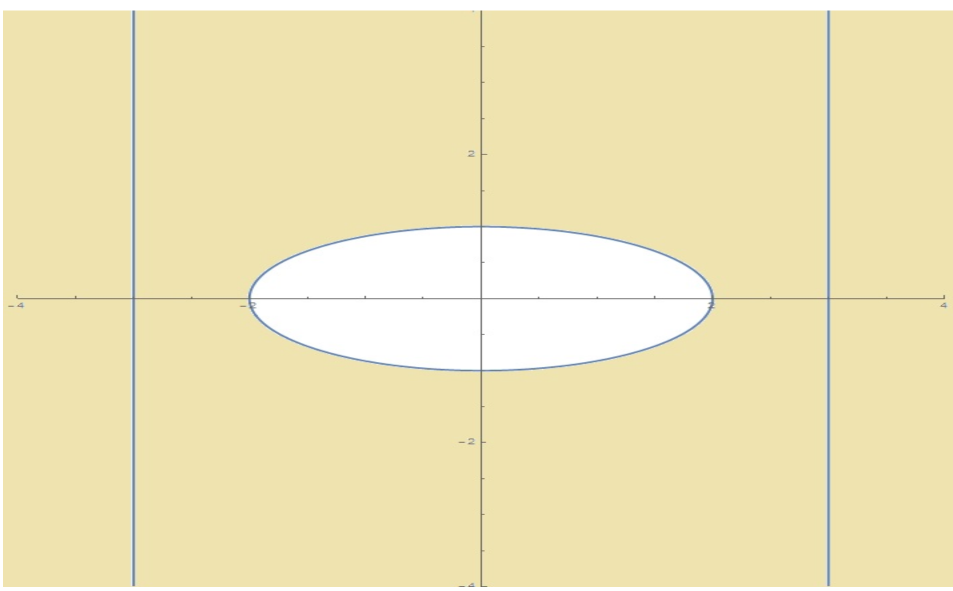
Name: Date: 09/27/2018

M20550 Calculus III Tutorial
Worksheet 4

1. Find and sketch the domain of the function
$$f(x, y) = \frac{\ln(x^2 + 4y^2 - 4)}{9 - x^2}.$$

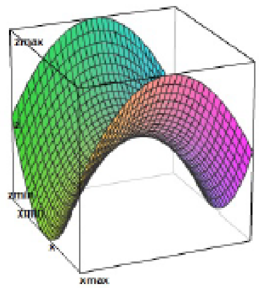
Solution: The domain of the function is the set of pairs (x, y) we can plug into the function. Since the function is a fraction, the denominator cannot be zero. Thus we have that
$$x^2 \neq 9 \Leftrightarrow x \neq \pm 3.$$
Furthermore, in the numerator, the input of ln must be positive, thus
$$x^2 + 4y^2 - 4 > 0 \Leftrightarrow x^2 + 4y^2 > 4.$$
So, as a set, the domain is
$$\text{domain}(f) = \left\{ (x, y) |\, x \neq \pm 3 \text{ and } x^2 + 4y^2 > 4 \right\}.$$
Graphically, $x \neq \pm 3$ removes the vertical lines $x = -3$ and $x = 3$ from the domain, and $x^2 + 4y^2 > 4$ or $\frac{x^2}{4} + y^2 > 1$ says that we can only take points outside of the ellipse $\frac{x^2}{4} + y^2 > 1$.



The yellow shaded region is the domain (the blue lines are removed from the domain).

2. Select the correct graph and the correct contour plot of level curves for the function

$$f(x, y) = x^2 - y^2$$

I.


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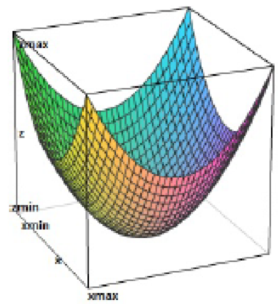

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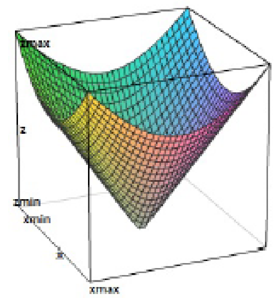
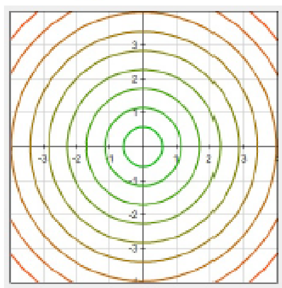
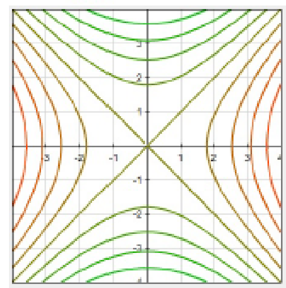
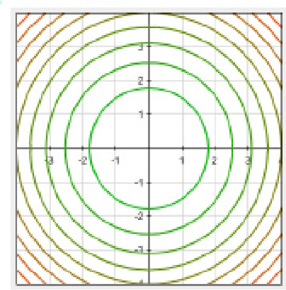


A.


B.


C.


(a) I and B (b) I and A (c) II and A

(d) II and C (e) III and C

Solution: Let's first determine the level curves for f. These are the family of curves with the equation $x^2 - y^2 = k$, where k is a constant. These equations are obviously not equations of circles. So, we eliminate choices A and C. Thus, B must be the level curves (or contour plot) for f. Here, $x^2 - y^2 = k$ with k is a constant give us a family of hyperbolas. Based upon the level curves for f, we easily see that the graph of f cannot be II or III. It must be I.

3. Evaluate the following limit

$$\lim_{(x,y)\to(0,0)} \frac{y + xe^{-y^2}}{1 + x^2}.$$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{y + xe^{-y^2}}{1 + x^2} = \frac{0 + 0 \cdot e^0}{1 + 0} = \frac{0}{1} = 0.$$

4. Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}.$$

Solution: Let $f(x,y) = \dfrac{x^2y}{x^4+y^2}$. We will show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by showing that f approaches two different values as (x,y) approaches $(0,0)$ along two different paths.

First, let (x,y) approach $(0,0)$ along the x-axis, i.e. $y=0$. We have $f(x,0) = \dfrac{0}{x^4+0} = 0$. So,

$$f(x,y) \longrightarrow 0 \text{ as } (x,y) \longrightarrow (0,0) \text{ along the } x\text{-axis.}$$

Next, let (x,y) approach $(0,0)$ along the curve $y = x^2$. We have $f(x,x^2) = \dfrac{x^2(x^2)}{x^4+(x^2)^2} = \dfrac{x^4}{2x^4} = \dfrac{1}{2}$. So,

$$f(x,y) \longrightarrow \frac{1}{2} \text{ as } (x,y) \longrightarrow (0,0) \text{ along the curve } y = x^2.$$

Since f admits two different limits along two different paths, the limit does not exist.

5. Find the second partial derivative g_{xy} of the function

$$g(x,y) = x^3y^2 + e^{xy}.$$

Solution: First, we find g_x by regarding y as a constant in $g(x,y) = x^3y^2 + e^{xy}$. We have,

$$g_x = 3x^2y^2 + ye^{xy}.$$

Now, $g_{xy} = (g_x)_y$. So, we regard x as a constant in $g_x = 3x^2y^2 + ye^{xy}$ and get

$$g_{xy} = 6x^2y + 1 \cdot e^{xy} + ye^{xy}(x).$$

or

$$g_{xy} = 6x^2y + e^{xy} + xye^{xy}.$$

6. Let $z = z(x,y)$ be defined implicitly as a function of x and y by the equation

$$x^2e^y = -z\cos(yz).$$

Find $\dfrac{\partial z}{\partial x}$ at the point $x = 1$, $y = 0$, and $z = -1$.

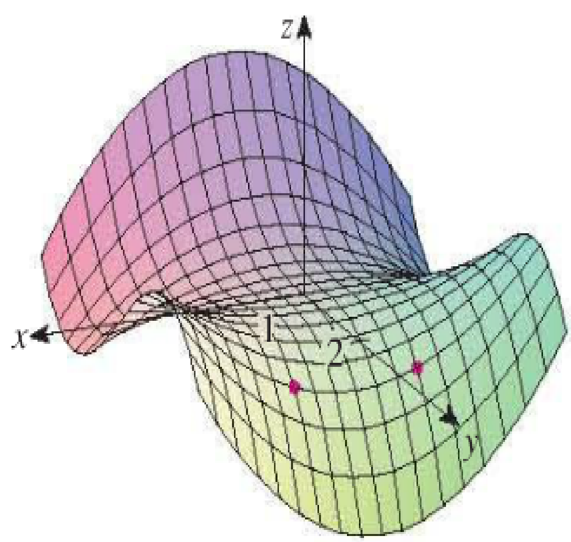
Solution: We're going to use implicit differentiation. And because we're looking for $\dfrac{\partial z}{\partial x}$, y is going to be treated as a constant. And here, z is understood to be a function of x. We have

$$\begin{aligned}\frac{\partial}{\partial x}\left[x^2 e^y\right] &= \frac{\partial}{\partial x}\left[-z\cos(yz)\right]\\ 2xe^y &= -1\frac{\partial z}{\partial x}\cos(yz) + -z\left(-\sin(yz)\right)\left(y\frac{\partial z}{\partial x}\right)\\ 2xe^y &= -\frac{\partial z}{\partial x}\cos(yz) + yz\sin(yz)\frac{\partial z}{\partial x} \qquad (*)\end{aligned}$$

We want to find $\dfrac{\partial z}{\partial x}$ at the point $x = 1$, $y = 0$, and $z = -1$. Plugging these values for x, y, z into equation $(*)$, we then find out the value of $\dfrac{\partial z}{\partial x}$:

$$2(1)e^0 = -\frac{\partial z}{\partial x}\cos(0) + 0 \implies 2 = -\frac{\partial z}{\partial x} \implies \frac{\partial z}{\partial x} = -2.$$

7. The graph of f is shown below



Determine the sign of

(a) $f_x(1, 2)$

(b) $f_y(1, 2)$

(c) $f_x(-1, 2)$

(d) $f_y(-1, 2)$

Solution:

(a) $f_x(1, 2)$ gives the rate of change of f as we move from the point $(1, 2)$ in positive x-direction with y being fixed. We see that f increases in this direction. So, $f_x(1, 2)$ is positive.

(b) $f_y(1, 2)$ gives the rate of change of f as we move from the point $(1, 2)$ in positive y-direction with x being fixed. We see that f decreases in this direction. So, $f_y(1, 2)$ is negative.

(c) $f_x(-1, 2)$ gives the rate of change of f as we move from the point $(-1, 2)$ in positive x-direction with y being fixed. We see that f decreases in this direction. So, $f_x(-1, 2)$ is negative.

(d) $f_y(-1, 2)$ gives the rate of change of f as we move from the point $(-1, 2)$ in positive y-direction with x being fixed. We see that f decreases in this direction. So, $f_y(-1, 2)$ is negative.

8. Let $f(x, y) = \ln(xy)$. Find the maximum rate of change of f at $(1, 2)$ and the direction in which it occurs.

Solution: It is a fact that f changes the fastest in the direction of its gradient vector and the maximum rate of change is the magnitude of the gradient vector.

With $f(x, y) = \ln(xy)$, we first compute $\nabla f(1, 2)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{y}{xy}, \frac{x}{xy} \right\rangle = \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle$$

$$\implies \nabla f(1, 2) = \left\langle 1, \frac{1}{2} \right\rangle.$$

$$\implies |\nabla f(1, 2)| = \left| \left\langle 1, \frac{1}{2} \right\rangle \right| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}.$$

So, the maximum rate of change of f at $(1, 2)$ is $\frac{\sqrt{5}}{2}$ and the direction in which it occurs is $\left\langle 1, \frac{1}{2} \right\rangle$.

9. Find all points on the surface $z = x^2 - y^3$ where the tangent plane is parallel to the plane $x + 3y + z = 0$.

Solution: First, rewrite $z = x^2 - y^3$ into the level surface $F(x, y, z) = x^2 - y^3 - z = 0$ then $\nabla F(x, y, z) = \langle 2x, -3y^2, -1\rangle$ gives a normal vector to the tangent plane at any point (x, y, z) on the surface.

We want to find a point (x, y, z) such that the tangent plane is parallel to the plane $x + 3y + z = 0$; so we want to find x, y, z such that $\nabla F(x, y, z) = k\langle 1, 3, 1\rangle$, for some scalar k. We have $\langle 2x, -3y^2, -1\rangle = k\langle 1, 3, 1\rangle$ implies

$$\begin{cases} 2x &= k \\ -3y^2 &= 3k \\ -1 &= k \end{cases}$$

So, $k = -1$ (no other k works for this system of equations). Thus, we get $2x = -1 \implies x = -\frac{1}{2}$, and $-3y^2 = -3 \implies y = \pm 1$. Now we need to find z. Remember the point (x, y, z) we are looking for is on the surface $z = x^2 - y^3$.

So then with $x = -\frac{1}{2}$ and $y = 1$, we get $z = \left(-\frac{1}{2}\right)^2 - (1)^3 = -\frac{3}{4}$.

And with $x = -\frac{1}{2}$ and $y = -1$, we get $z = \left(-\frac{1}{2}\right)^2 - (-1)^3 = \frac{5}{4}$.

So, at the points $\left(-\frac{1}{2}, 1, -\frac{3}{4}\right)$ and $\left(-\frac{1}{2}, -1, \frac{5}{4}\right)$, the tangent plane to the surface $z = x^2 - y^3$ is is parallel to the plane $x + 3y + z = 0$.

10. The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Use the geometry of partial derivative to find the **slope** for the tangent line to this parabola at the point $(1, 2, -4)$.

Solution: We have $z = f(x, y) = 6 - x - x^2 - 2y^2$. The vertical plane $x = 1$ intersects the graph of f in a parabola (the x-direction is fixed). Thus, $f_y(1, 2)$ gives the slope of the tangent line of this parabola at the point $(1, 2, -4)$. We have

$$f_y = -4y \implies f_y(1, 2) = -4(2) = -8.$$

Thus, the required slope is -8.