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M20550 Calculus III Tutorial
Worksheet 5

1. Let $f(x,y,z) = x^3 - y^2 z$. If $\mathbf{v} = \langle 1,0,1\rangle$, find the directional derivative of f in the direction of $\mathbf{v}$ at the point $(1,1,1)$. At what rate is f changing at the given point as we move in the direction of $\mathbf{v}$? Is f increasing or decreasing in this instance?

Solution: The directional derivative of f in the direction of $\mathbf{v}$ at the point $(1,1,1)$, denote $D_{\mathbf{u}}f(1,1,1)$ where $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$, is given by

$$D_{\mathbf{u}}f(1,1,1) = \nabla f(1,1,1) \cdot \mathbf{u}$$

First,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 1,0,1\rangle}{\sqrt{1^2+0^2+1^2}} = \frac{1}{\sqrt{2}}\langle 1,0,1\rangle .$$

Secondly, the gradient of f is given by:

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\ &= \langle 3x^2, -2yz, -y^2\rangle \\ \implies \nabla f(1,1,1) &= \langle 3,-2,-1\rangle. \end{aligned}$$

So, now

$$\begin{aligned} D_{\mathbf{u}}f(1,1,1) &= \nabla f(1,1,1) \cdot \mathbf{u} \\ &= \langle 3,-2,-1\rangle \cdot \frac{1}{\sqrt{2}}\langle 1,0,1\rangle \\ &= \frac{1}{\sqrt{2}}\langle 3,-2,-1\rangle \cdot \langle 1,0,1\rangle \\ &= \frac{1}{\sqrt{2}}(3-1) \\ &= \sqrt{2} \end{aligned}$$

At the point $\underline{(1,1,1)}$, the value of the function f is $\boxed{\text{increasing}}$ at the rate of $\underline{\sqrt{2}}$ as we move in the direction given by the vector $\underline{\langle 1,0,1\rangle}$.

2. Find the tangent plane and the normal line to the surface $x^2 + y^2 = 2z^2$ at the point $P = (1,1,1)$.

Solution: The given surface is the zero level surface of the function $F(x,y,z) = x^2 + y^2 - 2z^2$. So, the normal vector to the tangent plane at the point $P(1,1,1)$ is given by $\nabla F(1,1,1)$. We have

$$\nabla F(x,y,z) = \langle 2x, 2y, -4z\rangle \implies \nabla F(1,1,1) = \langle 2,2,-4\rangle.$$

Thus, the equation of the tangent plane at $(1,1,1)$ is

$$2(x-1) + 2(y-1) - 4(z-1) = 0 \implies x + y - 2z = 0,$$

and the equation for the normal line at $(1,1,1)$ is

$$\langle x,y,z\rangle = \langle 1,1,1\rangle + t\langle 2,2,-4\rangle = \langle 1+2t, 1+2t, 1-4t\rangle.$$

3. Write an equation of the tangent line to the curve of intersection between the two surfaces defined by $z = 2x^2 + y^2$ and $x^2 + 3y^2 + 2z^2 = 22$ at the point $(1,1,3)$.

   Hint: Think about the geometry of the gradient vectors. You don't have to parametrize the curve to do this problem.

Solution: The surface $z = 2x^2 + y^2$ can be written as the level surface $F(x,y,z) = 2x^2 + y^2 - z = 0$; and so the gradient of F is

$$\nabla F(x,y,z) = \langle 4x, 2y, -1\rangle .$$

Also, the gradient of the level surface $G(x,y,z) = x^2 + 3y^2 + 2z^2 = 22$ is

$$\nabla G(x,y,z) = \langle 2x, 6y, 4z\rangle .$$

The tangent vector at $(1,1,3)$ of the curve of intersection between these two surfaces is perpendicular to both vectors $\nabla F(1,1,3) = \langle 4,2,-1\rangle$ and $\nabla G(1,1,3) = \langle 2,6,12\rangle$. And

$$\nabla F(1,1,3) \times \nabla G(1,1,3) = \langle 4,2,-1\rangle \times \langle 2,6,12\rangle = \langle 30,-50,20\rangle .$$

Thus, $\langle 30,-50,20\rangle$ is a parallel vector of the tangent line to the curve of intersection at $(1,1,3)$. Thus, an equation of the required tangent line is

$$\langle x,y,z\rangle = \langle 1,1,3\rangle + t\,\langle 30,-50,20\rangle .$$

4. Find the local maximum and the local minimum value(s) and saddle point(s) of the function $z = 3x^2 - 6xy + 2y^3 + 1$.

Solution: First, let's find all the critical points of $z = 3x^2 - 6xy + 2y^3 + 1$:

$$\begin{cases} z_x(x,y) = 6x - 6y = 0 \implies y = x & (1) \\ z_y(x,y) = 6y^2 - 6x = 0 & (2) \end{cases}$$

With $y = x$, equation (2) becomes $6x^2 - 6x = 0 \implies 6x(x-1) = 0 \implies x = 0$ or $x = 1$. Thus, all the critical points are $(0,0)$ and $(1,1)$.

Now, we will use the Second Derivative Test to test each critical point. We want to compute

$$D(x,y) = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = z_{xx}z_{yy} - z_{xy}^2 = (6)(12y) - (-6)^2 = 72y - 36.$$

And we have

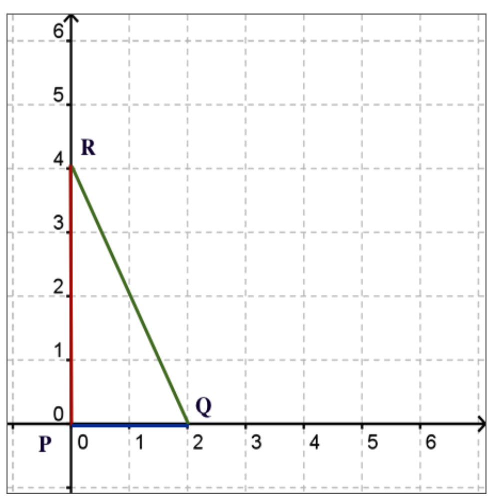
$$D(0,0) = -36 < 0 \implies (0,0) \text{ is a saddle point.}$$

$$D(1,1) = 72 - 36 = 36 > 0 \text{ and } z_{xx}(1,1) = 6 > 0 \implies z(1,1) \text{ is a local minimum.}$$

In conclusion, the local minimum value of z is $z(1,1) = 3(1)^2 - 6(1)(1) + 2(1)^3 + 1 = 0$. And $(0,0)$ is the saddle point of z, i.e. $z(0,0)$ is neither a local minimum nor local maximum.

5. Identify the absolute maximum and absolute minimum values attained by $g(x,y) = x^2y - x^2$ within the triangle T bounded by the points $P(0,0)$, $Q(2,0)$, and $R(0,4)$.

Solution: The picture for the triangle T:



First, we find all critical points in the interior of the triangle:

$$\begin{cases} g_x(x,y) = 2xy - 2x = 0 & (1) \\ g_y(x,y) = x^2 = 0 & (2) \end{cases}$$

Equation (2) tells us that x must be zero. And when $x = 0$, equation (1) is true automatically giving us the points $(0, y)$ for $0 \le y \le 4$ are the solutions of this system of equations. So, all the critical points are exactly the boundary PR of the triangle. So, we get no critical point inside the triangle. We move on to analyze the boundaries.

On the boundary PR, we have $x = 0$ and $0 \le y \le 4$. And, $g(0, y) = 0$.

On the boundary PQ, we have $0 \le x \le 2$ and $y = 0$. And, $g(x, 0) = -x^2$. The graph of $-x^2$ is a parabola concaves downward. So, $g(x, 0) = -x^2$ with $0 \le x \le 2$ attains a maximum value of 0 when $x = 0$ and a minimum value of -4 when $x = 2$.

On the boundary QR, we have $y = -2x + 4$ with $0 \le x \le 2$. And, $g(x, -2x + 4) = x^2(-2x + 4) - x^2 = -2x^3 + 3x^2$, for $0 \le x \le 2$. The critical numbers of $-2x^3 + 3x^2$ for $0 \le x \le 2$ are $x = 0$ and $x = 1$. So, g has a local minimum of 0 at $x = 0$, $y = 4$ and a local maximum of 1 at $x = 1$, $y = 2$ on this boundary, and $g(2, 0) = -4$ is the minimum on this boundary.

Here is a summary of the results:

(x,y)	$g(x,y)$
$(0,y)$	0
$(2,0)$	-4
$(1,2)$	1

So, we conclude that on the whole triangle (including boundaries), the function has an absolute maximum of 1 at $(1, 2)$ and an absolute minimum of -4 at $(2, 0)$.

6. Identify the absolute maximum and absolute minimum values attained by $z = xy + 1$ on the region $R = \{(x, y) \,|\, x^2 + y^2 \le 1\}$.

Solution: First, we find the critical points in the interior of the region R. We have

$$\begin{cases} z_x(x,y) = y = 0 & \implies y = 0 \\ z_y(x,y) = x = 0 & \implies x = 0 \end{cases}$$

So, the only critical point inside R is $(0, 0)$.

Next, we want to find the extreme values of z on the **boundary** $x^2 + y^2 = 1$. One way of doing this is to use the method of Lagrange Multipliers. In this language, we want to find the extrema of $z = xy+1$ subject to the constraint $g(x,y) = x^2+y^2 = 1$. We have $\nabla z = \lambda \nabla g$ for some constant λ. So, we get the system of equations:

$$\begin{cases} y = \lambda \cdot 2x & (1) \\ x = \lambda \cdot 2y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

Plug equation (2) into equation (1), we get $y = 4\lambda^2 y \Leftrightarrow (2\lambda+1)(2\lambda-1)y = 0 \implies y = 0$ or $\lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$.

- If $y = 0$, then from equation (2) we get $x = 0$. But $(x,y) = (0,0)$ contradicts equation (3). Thus, y cannot be zero, and we must have $\lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$.
- If $\lambda = \frac{1}{2}$, then from equation (1) or (2) we get $x = y$. With $x = y$, equation (3) gives $x = y = \pm\frac{\sqrt{2}}{2}$. So, the points of interest are $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.
- If $\lambda = -\frac{1}{2}$, then from equation (1) or (2) we get $x = -y$. With $x = -y$, equation (3) gives $x = -y = \pm\frac{\sqrt{2}}{2}$. So, the points of interest are $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Finally, let's compute the values of z at all the points we found:

(x,y)	$z = xy+1$
$(0,0)$	1
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\frac{3}{2}$
$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$\frac{3}{2}$
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$\frac{1}{2}$
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\frac{1}{2}$

In conclusion, the absolute maximum value of z is $\frac{3}{2}$ and it occurs at the points $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. The absolute minimum value of z is $\frac{1}{2}$ and it occurs at the points $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

7. Find the absolute maximum of $f(x,y,z) = xyz$ subject to the constraint

$x^2 + y^2 + 2z^2 = 9$, assuming that x, y, and z are nonnegative.

Solution: The gradient of f is

$$\nabla f = \langle yz, xz, xy \rangle .$$

Let $g = x^2 + y^2 + 2z^2$, then $\nabla g = \langle 2x, 2y, 4z \rangle$. The system of equations we get by Lagrange multipliers is thus

$$\begin{cases} yz &= 2\lambda x & \textcircled{1} \implies xyz = 2\lambda x^2 \\ xz &= 2\lambda y & \textcircled{2} \implies xyz = 2\lambda y^2 \\ xy &= 4\lambda z & \textcircled{3} \implies xyz = 4\lambda z^2 \\ x^2+y^2+2z^2 &= 9 & \textcircled{4} \end{cases}$$

Combining the first two new equations we get $2\lambda x^2 = 2\lambda y^2 \implies 2\lambda\left(x^2 - y^2\right) = 0$. So, either $\lambda = 0$ or $x^2 = y^2$.

Case 1: $\lambda = 0$. Then equation ① gives either $y = 0$ or $z = 0$. And we note that if either x, y, or z is zero, then f will be 0. So, we can move one from here and find other points and if 0 is the biggest value of f comparing to other points then 0 is an absolute maximum.

Case 2: $x^2 = y^2$

Similarly, combining the new second and third equations, we get $2\lambda y^2 = 4\lambda z^2 \implies 2\lambda\left(y^2 - 2z^2\right) = 0 \implies y^2 = 2z^2$ (we already considered the case when $\lambda = 0$).

So, we have in this case $x^2 = y^2$ and $y^2 = 2z^2 \implies x^2 = 2z^2$. Putting $y^2 = 2z^2$ and $x^2 = 2z^2$ into equation ④, we get $2z^2 + 2z^2 + 2z^2 = 9 \implies z = \sqrt{\frac{3}{2}}$ or $z = -\sqrt{\frac{3}{2}}$. According to the problem, we only consider the case where x, y, z are nonnegative.

With $z = \sqrt{\frac{3}{2}}$, $x^2 = 2z^2 \implies x^2 = 3 \implies x = \sqrt{3}$ $(x \geq 0)$.

And $y^2 = 2z^2 \implies y = \sqrt{3}$ $(y \geq 0)$. So, we get the point $\left(\sqrt{3}, \sqrt{3}, \sqrt{\dfrac{3}{2}}\right)$.

We have $f\left(\sqrt{3}, \sqrt{3}, \sqrt{\dfrac{3}{2}}\right) = \dfrac{3\sqrt{6}}{2}$ (which is bigger than 0 in case 1). Thus, the absolute maximum of f is $\dfrac{3\sqrt{6}}{2}$.

Optional/Review Problems:

8. (Chain Rule) If $h = x^2 + y^2 + z^2$ and $y\cos z + z\cos x = 0$, find $\dfrac{\partial h}{\partial x}$ assuming that x and y are the independent variables.

Solution: By definition we are finding how h changes when we vary x, but hold y constant and hold $y\cos z + z\cos x$ constant.

We have $h = h\big(x, y, z(x,y)\big)$. So,

$$\frac{\partial h}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \quad \text{since } z \text{ is a function of } x.$$

To find $\dfrac{\partial z}{\partial x}$, we use implicit differentiation:

$$\begin{aligned} y\cos z + z\cos x &= 0 \\ \frac{\partial}{\partial x}\big[y\cos z + z\cos x\big] &= \frac{\partial}{\partial x}[0] \\ -y\sin z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\cos x - z\sin x &= 0 \\ \frac{\partial z}{\partial x}(\cos x - y\sin z) &= z\sin x \\ \frac{\partial z}{\partial x} &= \frac{z\sin x}{\cos x - y\sin z} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial h}{\partial x} &= 2x + 2z\left(\frac{z\sin x}{\cos x - y\sin z}\right) \\ \implies \frac{\partial h}{\partial x} &= 2x + \frac{2z^2\sin x}{\cos x - y\sin z}. \end{aligned}$$

9. (Chain Rule) If $h = x^2 + y^2 + z^2$ and $y\cos z + z\cos x = 0$, find $\dfrac{\partial h}{\partial x}$ assuming that x and z are the independent variables.

Solution: We are finding how h changes when we vary x, but hold z constant and hold $y\cos z + z\cos x$ constant.

We have $h = h\big(x, y(x,z), z\big)$. So,

$$\frac{\partial h}{\partial x} = 2x + 2y\frac{\partial y}{\partial x} \quad \text{since } y \text{ is a function of } x.$$

To find $\frac{\partial y}{\partial x}$, use implicit differentiation:

$$y \cos z + z \cos x = 0$$
$$\frac{\partial}{\partial x}\left[y \cos z + z \cos x\right] = \frac{\partial}{\partial x}[0]$$

$$\frac{\partial y}{\partial x} \cos z - z \sin(x) = 0$$
$$\frac{\partial y}{\partial x} = z\frac{sin(x)}{cos(z)}$$

Therefore,

$$\frac{\partial h}{\partial x} = 2x + 2yz\frac{sin(x)}{\cos(z)}$$

The purpose of the last two questions was to get you to think about why the two $\partial z/\partial x$ are not the same. It is helpful to draw a picture with differentials to illustrate this. Talk to me in office hours/some other time if you have any questions.

10. (Chain Rule) Find $\frac{dz}{dt}$ when $t = 2$, where $z = x^2 + y^2 - 2xy$, $x = \ln(t-1)$ and $y = e^{-t}$.

Solution: We have $z = z\big(x(t), y(t)\big)$. So, by the chain rule, we obtain

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= (2x-2y)\left(\frac{1}{t-1}\right) + (2y-2x)e^{-t}(-1)\\ &= \left(2\ln(t-1) - 2e^{-t}\right)\left(\frac{1}{t-1}\right) - \left(2e^{-t} - 2\ln(t-1)\right)e^{-t}.\end{aligned}$$

Hence,

$$\begin{aligned}\left.\frac{dz}{dt}\right|_{t=2} &= \left(2\ln(2-1) - 2e^{-2}\right)\left(\frac{1}{2-1}\right) - \left(2e^{-2} - 2\ln(2-1)\right)e^{-2}\\ &= (0 - 2e^{-2})\cdot 1 - (2e^{-2} - 0)e^{-2}\\ &= -2e^{-2} - 2e^{-4}.\end{aligned}$$

11. (Chain Rule) Let $r = r(x, y)$, $x = x(s, t)$, and $y = y(t)$. Find $\frac{\partial r}{\partial t}$ at $(s, t) = (1, 0)$, given

$$\begin{array}{lll} x(1,0)=2, & x_s(1,0)=-1, & x_t(1,0)=7, \\ y(0)=3, & y(1)=0 & y'(0)=4, \\ r(2,3)=-1, & r_x(2,3)=3, & r_y(2,3)=5, \\ r_x(1,0)=6, & r_y(1,0)=-2, & \end{array}$$

Solution: We have $r = \big(x(s,t), y(t)\big)$. So, from the chain rule, we get

$$\begin{aligned} \frac{\partial r}{\partial t} &= \frac{\partial r}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial r}{\partial y}\frac{dy}{dt} \\ &= r_x x_t + r_y y' \\ &= r_x(x,y)x_t(s,t) + r_y(x,y)y'(t). \end{aligned}$$

When $s = 1$ and $t = 0$, we have $x = x(1,0) = 2$ and $y = y(0) = 3$. So,

$$\begin{aligned} \left.\frac{\partial r}{\partial t}\right|_{s=1,\,t=0} &= r_x(2,3)x_t(1,0) + r_y(2,3)y'(0) \\ &= (3)(7) + (5)(4) \\ &= 41. \end{aligned}$$

12. (Chain Rule) A cylinder containing an incompressible fluid is being squeezed from both ends. If the length of the cylinder is *decreasing* at a rate of 3m/s, calculate the rate at which the radius is changing when the radius is 2m and the length is 1m. (Note: An incompressible fluid is a fluid whose volume does not change.)

Solution: Let V be the volume of the cylinder, r be the radius of the cylinder, and l be its length. Then, $V = \pi r^2 l$. So, $V = V\big(r(t), l(t)\big)$.

By assumptions, we have $\dfrac{dl}{dt} = -3$ and incompressibility of the fluid implies $\dfrac{dV}{dt} = 0$.

We want to find $\dfrac{dr}{dt}$ at the instant when $r = 2$ and $l = 1$. We have

$$\begin{aligned} \frac{dV}{dt} &= \frac{d}{dt}\left[\pi r^2 l\right] \\ 0 &= 2\pi r l \frac{dr}{dt} + \pi r^2 \frac{dl}{dt}. \quad \text{And we know } \frac{dl}{dt} = -3; \text{ so} \\ 0 &= 2\pi r l \frac{dr}{dt} - 3\pi r^2 \\ \frac{dr}{dt} &= \frac{3r}{2l}. \end{aligned}$$

Hence, when $r = 2, l = 1$, we get $\dfrac{dr}{dt} = \dfrac{3 \cdot 2}{2 \cdot 1} = 3\text{m/s}$.

13. (Gradient) Find all the critical points of $f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$.

Solution: We want to find all points such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$. We have

$$\begin{cases} f_x(x,y) = 6xy - 12x = 0 & (1) \\ f_y(x,y) = 3y^2 + 3x^2 - 12y = 0 & (2) \end{cases}$$

Equation (1) implies $6x(y - 2) = 0 \implies x = 0$ or $y = 2$.

- When $x = 0$, equation (2) is equivalent to $3y^2 - 12y = 0 \implies 3y(y - 4) = 0 \implies y = 0$ or $y = 4$. So, we get the points $(0, 0)$ and $(0, 4)$.
- When $y = 2$, equation (2) is equivalent to $12 + 3x^2 - 24 = 0 \implies x^2 = 4 \implies x = -2$ or $x = 2$. So, we get the points $(-2, 2)$ and $(2, 2)$ here.

Thus, all the critical points of f are $(0, 0)$, $(0, 4)$, $(-2, 2)$, $(2, 2)$.

14. (Gradient) Find **all** points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

Solution: We know the direction of fastest change of f at a point (x, y) is given by the direction of $\nabla f(x, y) = \langle 2x - 2, 2y - 4\rangle$. So, we want to find all pairs (x, y) such that $\langle 2x - 2, 2y - 4\rangle = k\langle 1, 1\rangle$ for any constant k. We obtain the system of equations

$$\begin{cases} 2x - 2 & = k \\ 2y - 4 & = k \end{cases}$$

Then, $2x - 2 = 2y - 4 \implies y = x + 1$. Thus, all the wanted pairs (x, y) are $(x, x + 1)$, where x admits any value in the domain. This is exactly all the points on the line $y = x + 1$ in the domain of f.