Name: SOLUTIONS Date: 10/11/2018

# M20550 Calculus III Tutorial
## Worksheet 6

1. Evaluate the double integral $\iint_R (1-x)dA$, for $R = [0,1] \times [0,1]$, by identifying it as the volume of a solid.

> ***Solution:*** This integral is the volume of a solid whose base is $R = [0,1] \times [0,1]$ and whose height at any given $(x,y) \in [0,1] \times [0,1]$ is $(1-x)$. This solid is a triangular prism on its side. The volume is
>
> $$(\text{area of the triangle} \times \text{length of the prism}) = (0.5) \times 1 = 0.5$$
>
> So
>
> $$\iint_R (1-x)dA = 0.5$$

2. Evaluate the iterated integral.

   (a) $\int_0^2 \int_0^\pi r \sin^2\theta \; d\theta dr$

   > ***Solution:*** Since the region of integration is rectangular and the function is separable in $\theta$ and $r$, we can split it as a product of two integrals
   >
   > $$\int_0^2 r \; dr \cdot \int_0^\pi \sin^2\theta \; d\theta = 2 \cdot \int_0^\pi \frac{1}{2}(1-\cos 2\theta) \; d\theta = \pi$$

   (b) $\iint_R ye^{-xy} dA$ on $R = [0,2] \times [0,3]$

   > ***Solution:*** Notice that the region is rectangular, so the order of integration doesn't matter. However, we cannot separate this as a product of two integrals, since $x$ and $y$ are mixed variables in the function (we can't write it as a product of two functions $f(x)$ times $g(y)$).
   > We could try to integrate with respect to $y$ first, but that would require integration by parts. It turns out it is easier to start with $x$ instead:
   >
   > $$\int_0^3 \int_0^2 ye^{-xy} dxdy = \int_0^3 [-e^{-xy}]_{x=0}^{x=2} dy = \int_0^3 (-e^{-2y}+1)dy = \frac{1}{2}e^{-6} + \frac{5}{2}$$

3. Use polar coordinates to show that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dA = \pi$$

and deduce that $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$.

***Solution:*** We convert to polar coordinates, remembering that $dx\ dy$ becomes $r\ dr\ d\theta$. For the bounds, notice the original integral covers the entire plane. Thus we have

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\ dr\ d\theta$$

which now allows us to use $u$-substitution (which was impossible in the original integral). We take $u = r^2$, so that $du = 2r\ dr$. At the same time we may compute the integral over theta (which is $2\pi$), so we have

$$\pi\int_0^{\infty} e^{-u}du = \pi$$

Now, since the original integrand is a separable function of $x$ and $y$, i.e. it may be written as a product $e^{-x^2}e^{-y^2}$, and the region of integration is rectangular, our integrals are independent and we may write the original question as

$$\int_{-\infty}^{+\infty} e^{-x^2}dx \cdot \int_{-\infty}^{+\infty} e^{-y^2}dy$$

If we think of $y$ as a dummy variable, we notice that this is the integral we are trying to show equal to $\sqrt{\pi}$, times itself. This proves the desired result, since we have

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 = \pi$$

so

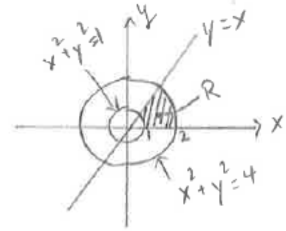
$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

4. Evaluate the given integral.

$$\iint_R \arctan\left(\frac{y}{x}\right)\ dA$$

where $R = \{(x, y) : 1 \le x^2 + y^2 \le 4, 0 \le y \le x\}$.

***Solution:***




Given the geometry of region R, it's best to compute the double integral using polar coordinates.

In polar, we know $dA = r\,dr\,d\theta$ and

$$\arctan\left(\frac{y}{x}\right) = \arctan(\tan\theta) = \theta \quad \left(\text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right).$$

From the picture of the region R, we have $1 \le r \le 2$. To find the upper bound for $\theta$, we need to find $\theta$ in (I) quad. such that $y = x$. With $y = x$, we have $r\sin\theta = r\cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$ for $\theta$ in (I) quad. So, $0 \le \theta \le \frac{\pi}{4}$.

Thus,

$$\iint_R \arctan\left(\frac{y}{x}\right) dA = \int_0^{\pi/4}\int_1^2 \theta\, r\,dr\,d\theta = \int_0^{\pi/4} \frac{1}{2}r^2\theta \Big|_{r=1}^{r=2} d\theta = \int_0^{\pi/4} \frac{3}{2}\theta\, d\theta = \frac{3}{2}\cdot\frac{1}{2}\theta^2\Big|_0^{\pi/4} = \boxed{\frac{3}{64}\pi^2}$$

5. Find the volume of the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

***Solution:*** Over any point $(r, \theta)$ inside the unit circle, there is $1 - (x^2 + y^2) = 1 - r^2$ of a volume over it. Hence the total volume is

$$\int_{\text{inside the unit circle}} (1 - r^2) dA = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} (1 - r^2)(r d\theta dr) =$$

$$2\pi \int_{r=0}^{1} (r - r^3) = 2\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)| = 2\pi * \frac{1}{4} = \pi/2$$

6. Set up, but do not evaluate, the integral that gives the volume of the solid region bounded by the paraboloid $z = x^2 + y^2$ and the cone $z = 1 - \sqrt{x^2 + y^2}$.

***Solution:*** The region of integration will be the interior of the projection of the curve of intersection of $z = x^2 + y^2$ with $z = 1 - \sqrt{x^2 + y^2}$. Setting the two equal to each other, we have

$$x^2 + y^2 = 1 - \sqrt{x^2 + y^2}$$

and due to the appearence of sums of $x^2$ and $y^2$, we choose to convert to polar coordinates. This choice is reinforced by the rotational symmetry of our solid along $z$-axis. Setting $x = r\cos\theta$ and $y = r\sin\theta$, the equation above becomes

$$r^2 = 1 - r$$

After rearranging as $r^2 + r - 1 = 0$, we find the only positive solution is $\frac{-1+\sqrt{5}}{2}$. Then our integral should be expressible as an integral over $\theta \in [0, 2\pi]$ and $r \in [0, \frac{-1+\sqrt{5}}{2}]$. We do top function (cone) minus bottom function (paraboloid), to get

$$\iint_R \left(1 - \sqrt{x^2+y^2} - (x^2+y^2)\right) dxdy = \int_0^{2\pi} \int_0^{\frac{-1+\sqrt{5}}{2}} (1 - r - r^2) r \; dr \; d\theta$$

7. (*1+1=2*) Prove the integration by parts formula

$$\int_0^a f(x)g(x)dx = f(a)\int_0^a g(y)dy - \int_{x=0}^a \frac{df}{dx}\int_{y=0}^x g(y)dydx$$

by changing the order of integration and using the fundamental theorem of calculus.

***Solution:*** We'll collapse the RHS:
The integral $\int_{x=0}^a \frac{df}{dx}\int_{y=0}^x g(y)dydx$ is over the area below the line $y = x$, above the x-axis, and bounded to the right by $x = a$. Hence

$$\int_{x=0}^a \frac{df}{dx}\int_{y=0}^x g(y)dydx = \int_{y=0}^a g(y)\int_{x=y}^a \frac{df}{dx}dxdy.$$

By the fundamental theorem of calculus this is $\int_{y=0}^a g(y)(f(a) - f(y))dy$

Hence

$$RHS = f(a)\int_0^a g(y)dy - \int_{y=0}^a g(y)(f(a) - f(y))dy =$$

$$f(a)\int_0^a g(y)dy + \int_{y=0}^a g(y)f(y)dy - f(a)\int_0^a g(y)dy = \int_{y=0}^a g(y)f(y)dy = LHS$$

Therefore QED.

8. (Optional) Find the maximum value of the function $f(x, y, z) = x + y$ on the curve of intersection of the plane $x + y + z = 1$ and the cylinder $y^2 + z^2 = 1$.

***Solution:*** Basically, the problem asks to maximize $f$ subject to two constraints:

$$g(x, y, z) = x + y + z = 1$$
$$h(x, y, z) = y^2 + z^2 = 1$$

We'll do this problem by the method of Lagrange Multipliers: First compute

$$\nabla f(x, y, z) = \langle 1, 1, 0\rangle$$
$$\nabla g(x, y, z) = \langle 1, 1, 1\rangle$$
$$\nabla h(x, y, z) = \langle 0, 2y, 2z\rangle$$

We know $\nabla f = \lambda \nabla g + \mu \nabla h$ for some scalars $\lambda$, $\mu$. So, along with the two constraints, we have the following system of equations:

$$\begin{cases} 1 &= \lambda & (1) \\ 1 &= \lambda + 2\mu y & (2) \\ 0 &= \lambda + 2\mu z & (3) \\ x + y + z &= 1 & (4) \\ y^2 + z^2 &= 1 & (5) \end{cases}$$

We get $\lambda = 1$ from equation (1). Putting this into equations (2) and (3), we get that $\mu y = 0$ and $\mu z = -\frac{1}{2}$. Hence we know $\mu \neq 0$. So $y = 0$. From (5) we know $z = \pm 1$. If $z = 1$ then from (4) we know $x = -1$. And if $z = -1$ then $x = 1$.

Hence our critical points along the constraints are $(-1, 0, 1)$ where $f = -1$ and $(1, 0, -1)$ where $f = 1$. Hence the maximum value of $f$ along the constraints is 1.

9. (Optional) The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on the ellipse that are nearest and farthest from the origin.

***Solution:*** We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ (this corresponds to distance function from origin squared) subject to the two constraints $g = x + y + 2z = 2$ and $h = x^2 + y^2 - z = 0$. Using the gradient equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

we obtain the system

$$\begin{cases} 2x = \lambda + 2\mu x \\ 2y = \lambda + 2\mu y \\ 2z = 2\lambda - \mu \\ x + y + 2z = 2 \\ x^2 + y^2 - z = 0 \end{cases}$$

Solving the equations, we obtain the points $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$. Then we have $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ (which is closest to the origin) and $f(-1, -1, 2) = 6$ (which is farthest from the origin).