Name: Date: 29/11/2018

M20550 Calculus III Tutorial
Worksheet 9

1. Determine whether or not the following vector fields are conservative:

   (a) $\mathbf{F} = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$

   (b) $\mathbf{F} = \mathbf{i} + \sin z\;\mathbf{j} + y\cos z\;\mathbf{k}$

> ***Solution:*** (a) Since $\mathbf{F}$ is a vector field on $\mathbb{R}^2$, we use the criterion $\frac{\partial P}{\partial y} \stackrel{?}{=} \frac{\partial Q}{\partial x}$ to see if $\mathbf{F}$ is conservative or not. We have $\mathbf{F} = \langle 3 + 2xy, x^2 - 3y^2\rangle$. So, $P = 3 + 2xy$ and $Q = x^2 - 3y^2$ and
>
> $$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}.$$
>
> Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^2$.
>
> (b) Since $\mathbf{F}$ is a vector field on $\mathbb{R}^3$, we use the criterion curl $\mathbf{F} \stackrel{?}{=} \mathbf{0}$ to see if $\mathbf{F}$ is conservative or not. We have $\mathbf{F} = \langle 1, \sin z, y\cos z\rangle$. And
>
> $$\text{curl }\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & \sin z & y\cos z \end{vmatrix} = \langle \cos z - \cos z, 0, 0\rangle = \langle 0,0,0\rangle = \mathbf{0}.$$
>
> Since curl $\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^3$.

2. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = -2xy\,\mathbf{i} + 4y\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2$.

> ***Solution:*** Since $x = t$, $y = t^2$, $z = 1$, we have
>
> $$\mathbf{F}(\mathbf{r}(t)) = -2t^3\mathbf{i} + 4t^2\mathbf{j} + \mathbf{k} = \langle -2t^3, 4t^2, 1\rangle,$$
>
> and
>
> $$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} = \langle 1, 2t, 0\rangle$$

The line integral of $\mathbf{F}$ along C is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt\\ &= \int_0^2 \langle -2t^3, 4t^2, 1\rangle \cdot \langle 1, 2t, 0\rangle\,dt\\ &= \int_0^2 \left(-2t^3 + 8t^3\right)\,dt\\ &= \int_0^2 6t^3\,dt\\ &= \left.\frac{6t^4}{4}\right|_0^2\\ &= \frac{3\cdot 2^4}{2} - 0\\ &= 24.\end{aligned}$$

Remark: Note that $\mathbf{F}$ is not a conservative vector field, so we cannot apply the Fundamental Theorem of Line Integrals in this example. To see this note that

$$\text{curl }\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ -2xy & 4y & 1\end{vmatrix} = \langle 0,0,2x\rangle \neq \mathbf{0}.$$

3. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = (y^2\cos(xy^2)+3x^2)\,\mathbf{i} + (2xy\cos(xy^2)+2y)\,\mathbf{j}$ is a conservative vector field and C is any curve from the point $(-1,0)$ to $(1,0)$.

Solution: Since we know $\mathbf{F}$ is a conservative vector field, $\mathbf{F} = \nabla f$ for some scalar function $f(x,y)$. So, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r}$. Then, by the fundamental theorem of line integral (FTLI), we have $\int_C \nabla f\cdot d\mathbf{r} = f(1,0) - f(-1,0)$. So, let's go about and find the potential function $f(x,y)$ of $\mathbf{F}$ first.

We know $\mathbf{F} = \nabla f$, so $\langle y^2\cos(xy^2)+3x^2, 2xy\cos(xy^2)+2y\rangle = \langle f_x, f_y\rangle$. Thus, we have

$$f_x = y^2\cos(xy^2) + 3x^2 \tag{1}$$
$$f_y = 2xy\cos(xy^2) + 2y \tag{2}$$

Using equation (1), we have $f = \int (y^2 \cos(xy^2) + 3x^2)\, dx = \sin(xy^2) + x^3 + g(y)$. Now, we need to find $g(y)$ to complete f.

With $f = \sin(xy^2) + x^3 + g(y)$, we compute $f_y = 2xy\cos(xy^2) + g'(y)$. Then from equation (2) above, we must have

$$2xy\cos(xy^2) + g'(y) = 2xy\cos(xy^2) + 2y \implies g'(y) = 2y \implies g(y) = y^2 + C.$$

We only need *a* potential function to apply FTLI, so we can pick $C = 0$. So, a potential function $f(x, y)$ of the vector field $\mathbf{F}$ is

$$f(x, y) = \sin(xy^2) + x^3 + y^2.$$

Finally,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} &\overset{\text{FTLI}}{=} f(1, 0) - f(-1, 0)\\ &= (\sin 0 + 1^3 + 0^2) - (\sin 0 + (-1)^3 + 0^2)\\ &= 2.\end{aligned}$$

4. Use Green's Theorem to evaluate

$$\int_C \left(-\frac{y^3}{3} + \sin x\right) dx + \left(\frac{x^3}{3} + y\right) dy,$$

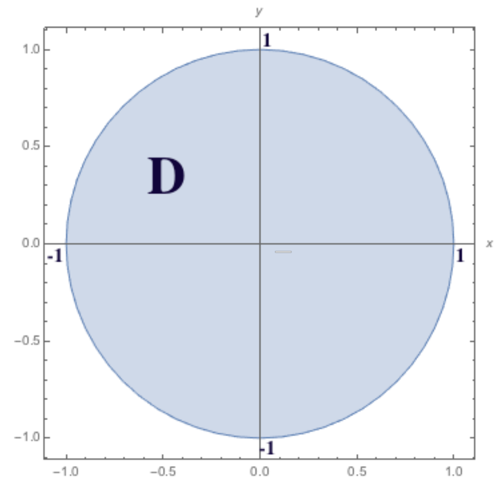
where C is the circle of radius 1 centered at $(0, 0)$ oriented counterclockwise when viewed from above.

Solution: Let D be the region enclosed by the unit circle C in this problem. By Green's Theorem, we have

$$\int_C \left(-\frac{y^3}{3} + \sin x\right) dx + \left(\frac{x^3}{3} + y\right) dy = \iint_D x^2 - (-y^2)\, dA.$$

$\left(\text{Here, we have } P = -\frac{y^3}{3} + \sin x \text{ and } Q = \frac{x^3}{3} + y, \text{ so } \frac{\partial P}{\partial y} = -y^2 \text{ and } \frac{\partial Q}{\partial x} = x^2.\right)$

So, instead of computing the line integral $\int_C \left(-\frac{y^3}{3} + \sin x\right) dx + \left(\frac{x^3}{3} + y\right) dy$, we are going to compute the double integral $\iint_D x^2 + y^2\, dA$, where D is the unit disk as shown below.



Using polar coordinates,

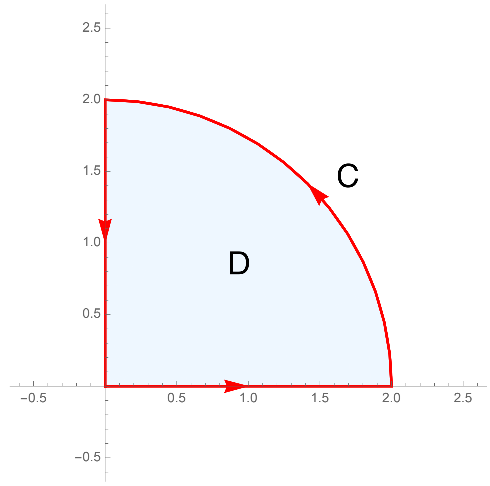
$$\iint_D x^2 + y^2 \, dA = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = 2\pi \left(\frac{1}{4}\right) = \frac{\pi}{2}.$$

Hence,

$$\int_C \left(-\frac{y^3}{3} + \sin x\right) dx + \left(\frac{x^3}{3} + y\right) dy = \frac{\pi}{2}.$$

5. A particle starts at the origin $(0,0)$, moves along the x-axis to $(2,0)$, then along the curve $y = \sqrt{4 - x^2}$ to the point $(0,2)$, and then along the y-axis back to the origin. Find the work done on this particle by the force field $\mathbf{F}(x,y) = y^2\,\mathbf{i} + 2x(y+1)\,\mathbf{j}$.

Solution: First we note that the curve C (drawn below) is a positively oriented, piecewise-smooth, simple closed curve in the plane. Let D be the region bounded by C.



The components of the vector field, $P = y^2$ and $Q = 2x(y+1)$, have continuous partial derivatives on an open region containing D (namely, all of $\mathbb{R}^2$). We may apply Green's Theorem:

$$\int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

Note that we have $\frac{\partial Q}{\partial x} = 2(y+1) = 2y+2$ and $\frac{\partial P}{\partial y} = 2y$. Finally, we compute the work done on the particle by the force field.

$$\begin{aligned} W = \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C y^2\,dx + 2x(y+1)\,dy \\ &\overset{Green}{=} \int\int_D (2y+2-2y)\;dA \\ &= 2\int\int_D\,dA \end{aligned}$$

Note that this is just twice the area of the region D. We may compute this as a double integral using polar coordinates $\left(W = 2\int_0^{\pi/2}\int_0^2 r\,dr\,d\theta\right)$ or by using the formula for the area of a circle. Thus,

$$W = 2(\text{Area of } D) = 2\left(\frac{\pi\cdot 2^2}{4}\right) = 2\pi.$$

6. (a) Compute div $\mathbf{F}$, where $\mathbf{F} = \langle e^y, zy, xy^2\rangle$.

   (b) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2\rangle$? Why?

Solution: (a) div $\mathbf{F} = \frac{\partial}{\partial x}(e^y) + \frac{\partial}{\partial y}(zy) + \frac{\partial}{\partial z}(xy^2) = 0 + z + 0 = z$

(b) For this problem, we need to remember the fact

$$\text{div curl } \mathbf{F} = 0 \quad \text{for any vector field } \mathbf{F}.$$

If there is a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2\rangle$ then by the fact above, $\mathbf{G}$ would satisfy the rule

$$\text{div curl } \mathbf{G} = 0 \quad \text{or} \quad \text{div } \left\langle xyz, -y^2z, yz^2 \right\rangle = 0.$$

But,

$$\text{div } \left\langle xyz, -y^2z, yz^2 \right\rangle = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(-y^2z) + \frac{\partial}{\partial z}(yz^2) = yz - 2yz + 2yz = yz \neq 0.$$

Thus, there is no such $\mathbf{G}$.

7. Parametrize the following surfaces:
   (a) Part of the cylinder $x^2 + y^2 = 9$ between $z = -1$ and $z = 2$.
   (b) Par of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant.
   (c) Part of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 1$

Solution: (a) We can let $x(u, v) = 3\cos u$ and $y(u, v) = 3\sin u$ and $z(u, v) = v$. The domain for (u, v) would then be $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 2$. So a parametrization is

$$\mathbf{r}(u, v) = \langle 3\cos u, 3\sin u, v\rangle, \quad \text{where } 0 \leq u \leq 2\pi \text{ and } -1 \leq v \leq 2.$$

(b) For this part of the sphere, we choose ϕ and θ to be the parameters. And since the radius of this sphere is 2, we have

$$x(\phi, \theta) = 2\sin\phi\cos\theta, \quad y(\phi, \theta) = 2\sin\phi\sin\theta, \quad z(\phi, \theta) = 2\cos\phi,$$

where the domain is $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq \frac{\pi}{2}$ since this part of the sphere is in the first octant. Thus, a parametrization is

$$\mathbf{r}(\phi, \theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle, \quad \text{where } 0 \leq \phi \leq \frac{\pi}{2} \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

(c) For this surface, we want to parametrize the paraboloid $z = x^2 + y^2$ with the condition that all the x's and y's must satisfy $x^2 + y^2 \leq 1$ (since we're looking at the part of the paraboloid **inside** the cylinder $x^2 + y^2 = 1$.) We can choose x and y to be the parameters, then we have

$$x(x,y) = x, \quad y(x,y) = y, \quad z(x,y) = x^2 + y^2, \quad \text{where} \quad x^2 + y^2 \leq 1.$$

So, a parametrization is

$$\mathbf{r}(x,y) = \left\langle x, y, x^2 + y^2 \right\rangle \quad \text{and the domain is} \quad x^2 + y^2 \leq 1.$$

8. Write an equation of the tangent plane to the parametric surface

$$x = u^2 + 1, \quad y = v^3 + 1, \quad z = u + v,$$

at the point $(5, 2, 3)$.

Solution: The surface is given by the vector equation $\mathbf{r}(u,v) = \langle u^2 + 1, v^3 + 1, u + v \rangle$. So, a normal vector to the tangent plane at $(5, 2, 3)$ is given by $\mathbf{r}_u \times \mathbf{r}_v$ at the point $(5, 2, 3)$.

First, $\mathbf{r}_u = \langle 2u, 0, 1 \rangle$ and $\mathbf{r}_v = \langle 0, 3v^2, 1 \rangle$. Now, we want to find (u, v) corresponds to the point $(x, y, z) = (5, 2, 3)$. So, we want to find (u, v) that satisfies:

$$5 = u^2 + 1, \quad 2 = v^3 + 1, \quad 3 = u + v.$$

$2 = v^3 + 1$ implies $v = 1$. So, $3 = u + v \implies 3 = u + 1 \implies u = 2$. And we see that $u = 2$ satisfies the equation $5 = u^2 + 1$. Thus, $(u, v) = (2, 1)$ gives the points $(x, y, z) = (5, 2, 3)$.

Now, with $u = 2$ and $v = 1$, we have $\mathbf{r}_u = \langle 4, 0, 1 \rangle$ and $\mathbf{r}_v = \langle 0, 3, 1 \rangle$. So, $\mathbf{r}_u \times \mathbf{r}_v = \langle 4, 0, 1 \rangle \times \langle 0, 3, 1 \rangle = \langle -3, -4, 12 \rangle$. So, $\langle -3, -4, 12 \rangle$ can be chosen as a normal vector to the tangent plane at the point $(5, 2, 3)$. And so, an equation of this tangent plane is given by

$$\langle -3, -4, 12 \rangle \cdot \langle x, y, z \rangle = \langle -3, -4, 12 \rangle \cdot \langle 5, 2, 3 \rangle$$
$$\implies -3x - 4y + 12z = 13.$$

9. Write the integral that computes the surface area of the surface S parametrized by $\mathbf{r}(u,v) = \langle u^2 \cos v, u^2 \sin v, v \rangle$, where $0 \leq u \leq 1$ and $0 \leq v \leq \pi$.

Solution: The area of the surface S is given by

$$\text{Area}(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

where D is the region given by $0 \leq u \leq 1$ and $0 \leq v \leq \pi$. With $\mathbf{r}(u,v) = \langle u^2 \cos v, u^2 \sin v, v\rangle$, we have $\mathbf{r}_u = \langle 2u \cos v, 2u \sin v, 0\rangle$ and $\mathbf{r}_v = \langle -u^2 \sin v, u^2 \cos v, 1\rangle$. Then

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2u \cos v, 2u \sin v, 0\rangle \times \left\langle -u^2 \sin v, u^2 \cos v, 1\right\rangle = \left\langle 2u \sin v, -2u \cos v, 2u^3\right\rangle.$$

So,

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= \left|\left\langle 2u \sin v, -2u \cos v, 2u^3\right\rangle\right| \\ &= \sqrt{(2u \sin v)^2 + (-2u \cos v)^2 + (2u^3)^2} \\ &= \sqrt{4u^2 + 4u^6} \\ &= \sqrt{4u^2(1+u^4)} = 2u\sqrt{1+u^4}. \end{aligned}$$

Finally,

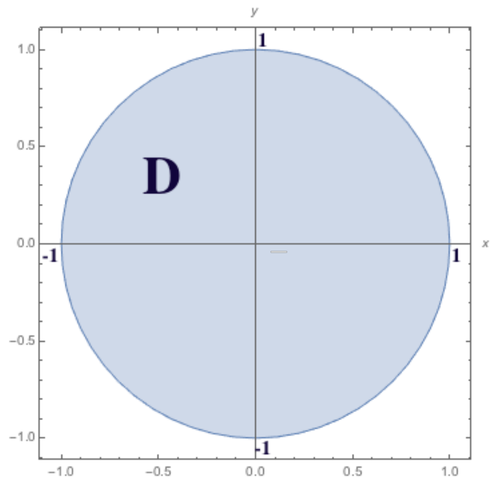
$$\text{Area}(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \iint_D 2u\sqrt{1+u^4} \, dA = \int_0^1 \int_0^\pi 2u\sqrt{1+u^4} \, dv \, du.$$

10. Find the area of the part of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 1$.

Solution: Denote S the surface given by the part of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2+y^2 = 1$. Since the surface S is given by the equation $z = x^2 + y^2$, we can use the following formula to compute the area of S:

$$\begin{aligned} \text{Area}(S) &= \iint_D \sqrt{1 + (z_x)^2 + (z_y)^2} \, dA \\ &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2} \, dA \\ &= \iint_D \sqrt{1 + 4(x^2 + y^2)} \, dA. \end{aligned}$$

Here, D is the projection of S onto the xy-plane. So, D is the unit disk in the xy-plane.



We use polar coordinate to compute the double integral above.

$$\begin{aligned}\iint_D \sqrt{1+4(x^2+y^2)}\, dA &= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{8}\left(\frac{2}{3}\right)\left(1+4r^2\right)^{3/2}\Big|_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \frac{1}{12}\left(5^{3/2}-1\right) d\theta \\ &= \frac{\pi}{6}\left(5^{3/2}-1\right).\end{aligned}$$

So, the area of the given surface is $\frac{\pi}{6}\left(5^{3/2}-1\right)$.

Alternatively, if you don't want to remember two formulas for surface area. You can still do this problem by using the formula

$$\text{Area}(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

In this case, we need a parametrization of S. Since the surface is given by the paraboloid $z = x^2 + y^2$, we can let x and y be the parameters and have $z = x^2 + y^2$. But the surface lies inside the cylinder $x^2 + y^2 = 1$, so x and y lie inside the unit disk $x^2 + y^2 \leq 1$ in the xy-plane. So, a parametrization of S is given by

$$\mathbf{r}(x,y) = \left\langle x, y, x^2+y^2 \right\rangle, \text{ for } (x,y) \in D,$$

where D is the disk centered at $(0,0)$ with radius 1 in the xy-plane as shown in the picture above.

Then, $\mathbf{r}_x = \langle 1, 0, 2x\rangle$ and $\mathbf{r}_y = \langle 0, 1, 2y\rangle$. So, $\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -2y, 1\rangle$. Then, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{4x^2+4y^2+1} = \sqrt{1+4(x^2+y^2)}$. And so,

$$\text{Area}(S) = \iint_D |\mathbf{r}_x \times \mathbf{r}_y|\, dA = \iint_D \sqrt{1+4(x^2+y^2)}\, dA = \frac{\pi}{6}\left(5^{3/2}-1\right) \text{ (as above).}$$