Multiple Choice

1.(7pts) Find the row reduced echelon form of the matrix $\begin{bmatrix} 2 & -1 & 7 & 3 \\ -1 & 2 & -8 & -3 \\ 1 & -1 & 5 & 2 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.(7pts) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Which of the following is equal to $(A+B)(A-B)$?

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3.(7pts) Which one of the following sets is a basis of $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

4.(7pts) Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ and $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformations such that

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \qquad S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Which matrix below is the standard matrix of ST ?

(a) $\begin{bmatrix} 2 & 0 & 2 \\ 0 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

5.(7pts) Let A denote the 3×3 matrix $A = \begin{bmatrix} 2 & 2 & 3 \\ 6 & 3 & 3 \\ 2 & 4 & 9 \end{bmatrix}$. Which of the following numbers is the value of the determinant $\det(A)$?

(a) 6 (b) -6 (c) 0 (d) 12 (e) -12

6.(7pts) The dimension of the null space of a 7×9 matrix B is 4. How many rows of zeros does the row reduced echelon form of B contain?

(a) 2 (b) 1 (c) 5 (d) 4 (e) 3

7.(7pts) Let $\mathcal{B}$ denote the basis of $\mathbb{R}^3$ given by $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v}$ denote the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Which of the following is the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to $\mathcal{B}$?

(a) $\begin{bmatrix} -1/3 \\ 1/2 \\ 1/6 \end{bmatrix}$

(b) $\begin{bmatrix} -1/3 \\ 1/2 \\ -1/6 \end{bmatrix}$

(c) $\begin{bmatrix} 1/3 \\ -1/2 \\ -1/6 \end{bmatrix}$

(d) $\begin{bmatrix} -1/3 \\ -1/2 \\ 1/6 \end{bmatrix}$

(e) $\begin{bmatrix} 1/3 \\ 1/2 \\ -1/6 \end{bmatrix}$

8.(7pts) Let A be an $n \times n$ square matrix. Suppose that for some $\mathbf{b}$ in $\mathbb{R}^n$, the linear system $A\mathbf{x} = \mathbf{b}$ has at least two (distinct) solutions. Which of the following statements must be true?

(a) The linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is onto.

(b) A has a pivot in every row.

(c) There is an $n \times n$-matrix B with $BA = \mathrm{I}_n$.

(d) The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The linear system $A\mathbf{x} = \mathbf{c}$ is inconsistent for some $\mathbf{c}$ in $\mathbb{R}^n$.

9.(7pts) Which of the following is the solution of the matrix equation $\begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$?

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7/3 & -2 \\ -4/3 & 1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$ (c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7/3 & -2 \\ -4/3 & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7/3 & 2 \\ 4/3 & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$ (e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

Partial Credit

10.(7pts) Which of the following is equal to value of the matrix product $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

(a) 8 (b) 4 (c) 9 (d) 6 (e) 2

11.(7pts) Let A denote the 3×3 matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 4 & 9 \end{bmatrix}$. Which of the following numbers is the value of the determinant $\det(A)$?

(a) -2 (b) 1 (c) 2 (d) 0 (e) -1

12.(7pts) The dimension of the null space of a 7×8 matrix B is 4. How many rows of zeros does the row reduced echelon form of B contain?

(a) 5 (b) 2 (c) 3 (d) 4 (e) 1

13.(7pts) Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ and $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformations such that

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \qquad S\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\-1\end{bmatrix} \text{ and}$$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}.$$

Which matrix below is the standard matrix of TS ?

(a) $\begin{bmatrix}2 & 0 & -2\\0 & 2 & 0\\2 & 0 & -2\end{bmatrix}$ (b) $\begin{bmatrix}2 & 0 & -2\\0 & -2 & 0\\2 & 0 & -2\end{bmatrix}$ (c) $\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$

(d) $\begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}$ (e) $\begin{bmatrix}2 & 0 & 2\\0 & -2 & 0\\-2 & 0 & -2\end{bmatrix}$

14.(7pts) Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by counterclockwise rotation of the plane about the origin by an angle of $\frac{\pi}{9}$ (in radians). Let A be the standard matrix of T. Which of the following matrices is equal to A^3 ?

(a) $\begin{bmatrix}1/2 & \sqrt{3}/2\\\sqrt{3}/2 & -1/2\end{bmatrix}$ (b) $\begin{bmatrix}1/2 & -\sqrt{3}/2\\-\sqrt{3}/2 & -1/2\end{bmatrix}$ (c) $\begin{bmatrix}1/2 & \sqrt{3}/2\\\sqrt{3}/2 & 1/2\end{bmatrix}$

(d) $\begin{bmatrix}1/2 & -\sqrt{3}/2\\\sqrt{3}/2 & 1/2\end{bmatrix}$ (e) $\begin{bmatrix}-1/2 & -\sqrt{3}/2\\\sqrt{3}/2 & -1/2\end{bmatrix}$

15.(12pts) Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 4 & 0 \\ -3 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$.

16.(12pts) Express the solution set of

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 4x_2 & - & 3x_3 & - & 12x_4 & = & 17 \\ x_1 & + & 2x_2 & - & 3x_3 & - & 9x_4 & = & 13 \\ x_1 & + & 2x_2 & - & 2x_3 & - & 7x_4 & = & 10 \end{array}$$

in *parametric vector form.*

17.(12pts) The row-reduced echelon form of the 3×5 matrix $A = \begin{bmatrix} 1 & 1 & -1 & 1 & 2 \\ 1 & 2 & -4 & 3 & 5 \\ 1 & 4 & -10 & 9 & 15 \end{bmatrix}$ is given by $B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. (You may assume this; you do not have to check it.)

(a) Determine a basis for the null space $\mathrm{Nul}(A)$.

(b) Determine a basis for the column space $\mathrm{Col}(A)$.

1. Solution. $\begin{bmatrix} 2 & -1 & 7 & 3 \\ -1 & 2 & -8 & -3 \\ 1 & -1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 5 & 2 \\ -1 & 2 & -8 & -3 \\ 2 & -1 & 7 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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2. Solution. $(A+B)(A-B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

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3. Solution. The vectors in (a) are the columns of an (invertible) upper triangular 3×3-matrix, so form a basis of $\mathbb{R}^3$.

(b) Two vectors can't span $\mathbb{R}^3$.

(c) Four vectors in $\mathbb{R}^3$ must be linearly dependent.

(d) Contains $\mathbf{0}$, so the set is linearly dependent.

(e) Second vector is a scalar multiple (-1) of the third, so the set is linearly dependent.

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4. Solution. The standard matrix of S is $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$ and that of T is $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

Hence the standard matrix of ST is the matrix product $AB = \begin{bmatrix} 2 & 0 & 2 \\ 0 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$.

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5. Solution. $\begin{vmatrix} 2 & 2 & 3 \\ 6 & 3 & 3 \\ 2 & 4 & 9 \end{vmatrix} = 3 \begin{vmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 2 & 4 & 9 \end{vmatrix} = 3 \cdot 2 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 4 & 9 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 2 & 6 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{vmatrix} =$ $6 \cdot 1 \cdot (-1) \cdot 2 = -12$ since the last matrix is upper triangular.

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6. Solution. $\operatorname{rank}(B) = 9 - \dim(\operatorname{Nul}(B)) = 9 - 4 = 5$. The reduced echelon form of B has 5 pivot rows and 7 rows altogether, so there are $7 - 5 = 2$ rows of zeros.

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7. Solution. We have to solve the linear system with augmented matrix the first matrix in: $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & -3 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & -6 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -6 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/6 \end{bmatrix}$

The coordinate vector is $\begin{bmatrix} 1/3 \\ 1/2 \\ -1/6 \end{bmatrix}$.

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8. Solution. Since $A\mathbf{x} = \mathbf{b}$ has more than one solution, the linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is not one-to-one. The equivalent conditions for matrix invertibility show that A is not invertible, and T is not onto either, so $A\mathbf{x} = \mathbf{c}$ is inconsistent for some $\mathbf{c}$ in $\mathbb{R}^n$. Those conditions also show that the other listed conditions are all equivalent to invertibility of A, so cannot hold.

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9. Solution. The equation is $A\mathbf{v} = \mathbf{b}$ where $A = \begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} h \\ k \end{bmatrix}$. Note $\det A = 3 \cdot 7 - 4 \cdot 6 = -3$ so A is invertible and $A^{-1} = \frac{1}{-3}\begin{bmatrix} 7 & -6 \\ -4 & 3 \end{bmatrix}$. The solution is $\mathbf{v} = A^{-1}\mathbf{b}$ i.e. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7/3 & 2 \\ 4/3 & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$.

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10. Solution. $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 8.$

11. Solution. $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{vmatrix} = 1 \cdot (-1) \cdot 2 = -2$ since the last matrix is upper triangular.

12. Solution. $\text{rank}(B) = 8 - \dim(\text{Nul}(B)) = 8 - 4 = 4$. The reduced echelon form of B has 4 pivot rows and 7 rows altogether, so there are $7 - 4 = 3$ rows of zeros.

13. Solution. The standard matrix of S is $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$ and that of T is $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

Hence the standard matrix of TS is the matrix product $BA = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

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14. Solution. Note that A^3 is the matrix of the linear transformation $T = S^3$, which is the composite of three successive counterclockwise rotations of the plane about the origin by an angle of $\frac{\pi}{9}$. So T is the rotation of the plane about the origin by an angle of $3 \cdot \frac{\pi}{9} = \frac{\pi}{3}$. The matrix of T is $\begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

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15. Solution. Row-reduce $\begin{bmatrix} 1 & 4 & 0 & 1 & 0 & 0 \\ -3 & 2 & 3 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 14 & 3 & 3 & 1 & 0 \\ 0 & 4 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -3 \\ 0 & 4 & 1 & 1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 6 \\ 0 & 2 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 6 \\ 0 & 1 & 0 & 0 & 1/2 & -3/2 \\ 0 & 0 & 1 & 1 & -2 & 7 \end{bmatrix}$.

The inverse is $\begin{bmatrix} 1 & -2 & 6 \\ 0 & 1/2 & -3/2 \\ 1 & -2 & 7 \end{bmatrix}$.

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16. Solution. Row reduce: $\begin{bmatrix} 2 & 4 & -3 & -12 & 17 \\ 1 & 2 & -3 & -9 & 13 \\ 1 & 2 & -2 & -7 & 10 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 & -3 & 4 \\ 1 & 2 & -3 & -9 & 13 \\ 1 & 2 & -2 & -7 & 10 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 & -3 & 4 \\ 0 & 0 & -3 & -6 & 9 \\ 0 & 0 & -2 & -4 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 & -3 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & -2 & -4 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 & -3 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The equations are equivalent to

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & & & - & 3x_4 & = & 4 \\ & & & & x_3 & + & 2x_4 & = & -3 \end{array}$$

The bound variables are x_1, x_3, and free variables are x_2, x_4. Rewriting with free variables on the right,

$$\begin{array}{ccccccccc} x_1 & & = & 4 & - & 2x_2 & + & 3x_4 \\ & x_3 & = & -3 & & & - & 2x_4 \end{array}$$

In parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

or writing $x_2 = r$, $x_4 = s$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

17. Solution. (a) The pivot columns of A and B are 1, 2, 4, so x_3 and x_5 are free variables. Writing the homogeneous equations from B with free variables on the right gives

$$\begin{array}{rcrcr} x_1 & = & -2x_3 & - & x_5 \\ x_2 & = & 3x_3 & + & x_5 \\ x_4 & = & & & -2x_5 \end{array}$$

The system has 2 basic solutions given by setting one free variable equal to 1 and the others equal to 0. Setting $x_3 = 1$ and $x_5 = 0$ gives the basic solution $\mathbf{v_1} = \begin{bmatrix} -2 & 3 & 1 & 0 & 0 \end{bmatrix}^T$. Setting $x_3 = 0$ and $x_5 = 1$ gives the basic solution $\mathbf{v_2} = \begin{bmatrix} -1 & 1 & 0 & -2 & 1 \end{bmatrix}^T$. Then $\{ \mathbf{v_1}, \mathbf{v_2} \}$ is a basis for $\mathrm{Nul}(A)$.

(b) Row operations don't change the solution space of the homogeneous equation or the linear dependences of columns of a matrix. The pivot columns (1st, 2rd, 4th) of B form a basis for $\mathrm{Col}(B)$ so the pivot columns (1st, 2rd, 4th) of A form a basis for $\mathrm{Col}(A)$. A basis of $\mathrm{Col}(A)$ is given by $\{ \mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3} \}$ where $\mathbf{w_1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $\mathbf{w_2} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$ and $\mathbf{w_3} = \begin{bmatrix} 1 & 3 & 9 \end{bmatrix}^T$.
