Part I: Multiple choice questions (7 points each)

1. Find the closest point to $\begin{bmatrix}1\\1\\2\end{bmatrix}$ in the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\2\end{bmatrix}$.

(a) $\begin{bmatrix}-2\\1\\1\end{bmatrix}$ (b) $\begin{bmatrix}-1\\1\\2\end{bmatrix}$ (c) $\begin{bmatrix}1\\1\\2\end{bmatrix}$ (d) $\begin{bmatrix}8/5\\1\\6/5\end{bmatrix}$ (e) $\begin{bmatrix}-3/5\\1\\6/5\end{bmatrix}$

$$W = \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}\right\} = \text{Span}\left\{\underset{\vec{v_1}}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{\vec{v_2}}{\begin{bmatrix}-1\\0\\2\end{bmatrix}}\right\} \qquad \vec{x} = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

orthogonal basis

$$\text{proj}_W \vec{x} = \frac{\vec{x}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} + \frac{\vec{x}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} = \frac{1}{1}\vec{v_1} + \frac{3}{5}\vec{v_2} = \begin{bmatrix}-3/5\\1\\6/5\end{bmatrix}$$

2. Which of the following is a least square solution $\hat{\mathbf{x}}$ to the equation

$$\underset{A}{\begin{bmatrix}1 & -2\\2 & 1\\1 & -2\\2 & 1\end{bmatrix}} \mathbf{x} = \underset{\vec{b}}{\begin{bmatrix}1\\3\\1\\3\end{bmatrix}} ?$$

(a) $\begin{bmatrix}11/9\\1/9\end{bmatrix}$ (b) $\begin{bmatrix}3/2\\1/2\end{bmatrix}$ (c) $\begin{bmatrix}7/5\\1/5\end{bmatrix}$ (d) $\begin{bmatrix}1\\-2\end{bmatrix}$ (e) $\begin{bmatrix}2\\1\end{bmatrix}$

$$A^TA = \begin{bmatrix}10 & 0\\0 & 10\end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix}14\\2\end{bmatrix}$$

Normal equation: $A^TA\vec{x} = A^T\vec{b}$

$$\left[\begin{array}{cc|c}10 & 0 & 14\\0 & 10 & 2\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & 0 & 14/10\\0 & 1 & 2/10\end{array}\right]$$

So $\hat{x} = \begin{bmatrix}7/5\\1/5\end{bmatrix}$

3. Which of the following functions is a solution to the initial value problem

$$\frac{dy}{dt} = (y-t)^2 + 1; \qquad y(0) = -1?$$

(a) $y = \frac{1}{t+1} - 2$ (b) $y = t$ (c) $y = \frac{-1}{t+1} + t$

(d) $y = t - 1$ (e) $y = \frac{-2}{t+1} + 1$

Trial and error: eliminate (b) since $y(0) = 0 \neq -1$

try (a), (c), (d), (e).

(c): $\frac{dy}{dt} = \frac{1}{(t+1)^2} + 1$ $\Rightarrow \frac{dy}{dt} = (y-t)^2 + 1.$

$(y-t)^2 = \left(\frac{-1}{t+1}\right)^2 = \frac{1}{(t+1)^2}$

4. Let A be an $m \times n$ matrix. Which of the following may be *false*?

(a) The equation $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent for any $\mathbf{b}$ in $\mathbb{R}^m$.

(b) $A^T A$ is invertible.

(c) A solution to $A^T A\mathbf{x} = A^T\mathbf{b}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$.

(d) The columns of A^T lie in the column space of $A^T A$.

(e) If $A^T A\mathbf{x} = A^T\mathbf{b}$ then $A\mathbf{x} - \mathbf{b}$ is orthogonal to Col(A).

(b) may be false

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ not invertible.

5. Which of the following is a general solution to the differential equation

$$1 + (\frac{x}{y} - \sin y)\frac{dy}{dx} = 0?$$

(a) $xy + y\sin y - \sin y = c$ (b) $xy + y\cos y - \sin y = cy$
(c) $xy + y\sin y - \cos y = c$ (d) $xy + y\cos y - \sin y = c$
(e) $xy + y\cos y - \cos y = c$

$y\,dx + (x - y\sin y)\,dy = 0$ is exact

Find $f(x,y)$ with $\frac{\partial f}{\partial x} = y \Rightarrow f = xy + g(y)$

$\frac{\partial f}{\partial y} = x - y\sin y \Rightarrow g'(y) = -y\sin y$

$\Rightarrow g(y) = \int -y\sin y\,dy = y\cos y - \sin y + C$

6. Consider the initial value problem

$$\sin(2x) + \cos(3y)\frac{dy}{dx} = 0 \qquad y(\pi/2) = \pi/3$$

Which of the following implicitly defines the solution?

(a) $\frac{-\cos(2x)}{2} + \frac{\sin(3y)}{3} = \frac{-1}{2}$ (b) $-\cos(2x) + \sin(3y) = \frac{1}{2}$
(c) $\sin(2x) + \cos(3y) = 1$ (d) $-\cos(2x) + \sin(3y) = \frac{-1}{2}$
(e) $\frac{-\cos(2x)}{2} + \frac{\sin(3y)}{3} = \frac{1}{2}$

$\int \cos(3y)\,dy = \int (-\sin 2x)\,dx$

$\frac{\sin(3y)}{3} = \frac{\cos(2x)}{2} + C$

$x = \frac{\pi}{2}$ $\rightsquigarrow$ $y(\frac{\pi}{2}) = \frac{\pi}{3}$

$\frac{\sin \pi}{3} = \frac{\cos \pi}{2} + C$

$0 = \frac{-1}{2} + C$

$\boxed{C = \frac{1}{2}}$

$\frac{\sin(3y)}{3} = \frac{\cos(2x)}{2} + \frac{1}{2}$

7. Let $y(t)$ be the unique solution of the initial value problem

$$(t^2 - t)\frac{dy}{dt} + \cos(\pi t)y = \frac{t^2 - t}{t - 2} \qquad y(3/2) = 0$$

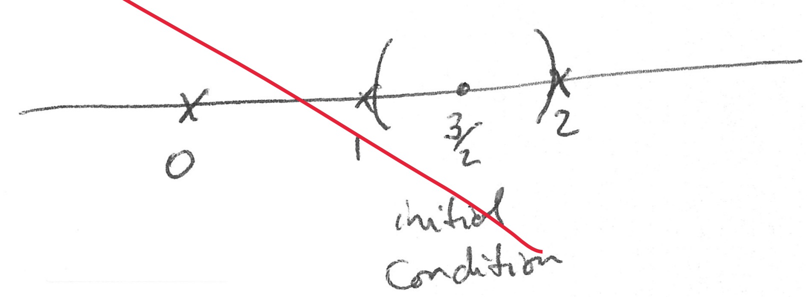
What is the largest interval where y is defined?

(a) $t > 0$ (b) $0 < t < 2$ (c) $1 < t < 2$ (d) $t < 1/2$ (e) $t < 2$

$$\frac{dy}{dt} + \frac{\cos(\pi t)}{t^2 - t}\, y = \frac{1}{t-2}$$

$t = 0, 1$ bad

$t = 2$ bad



8. A tank initially contains $100l$ of pure water. Then, at $t = 0$, a sugar solution with concentration of $4g/l$ starts being pumped into the tank at a rate of $5l/\text{min}$. The tank is kept well mixed, and the solution is being pumped out at the rate of $4l/\text{min}$. Which of the following is the initial value problem for $y(t)$ = quantity of sugar, in grams, in the tank at time t?

(a) $\dfrac{dy}{dt} = 5y - 4(100 + t)$ $\quad y(0) = 0$

(b) $\dfrac{dy}{dt} = 20 - 4y$ $\quad y(0) = 0$

(c) $\dfrac{dy}{dt} = 4$ $\quad y(0) = 100$

(d) $\dfrac{dy}{dt} = 20 - \dfrac{4y}{100 + t}$ $\quad y(0) = 0$

(e) $\dfrac{dy}{dt} = 20 - \dfrac{y}{(100 + t)^2}$ $\quad y(0) = 100$

tank at time t has

$100 + 5t - 4t = 100 + t$

liters of mixture

$$\frac{dy}{dt} = \underbrace{4 \cdot 5}_{\text{going in}} - \underbrace{\frac{y}{100+t} \cdot 4}_{\text{going out}}$$

$y(0) = 0$ no sugar at time 0

Part II: Partial credit questions (11 points each). Show your work.

9. Using the Gram-Schmidt Process, find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\2\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\3\\1\\3\end{bmatrix}$.

$$\underset{\vec{x}_1}{\parallel} \quad \underset{\vec{x}_2}{\parallel} \quad \underset{\vec{x}_3}{\parallel}$$

$\vec{v}_1 = \vec{x}_1$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$= \begin{bmatrix}1\\2\\1\\2\end{bmatrix} - \frac{1}{1}\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\2\\1\\2\end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix}1\\3\\1\\3\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{13}{9}\begin{bmatrix}0\\2\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1/9\\-4/9\\1/9\end{bmatrix}$$

$\boxed{\text{Orthogonal}}$

$\vec{v}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\quad \underline{\underline{\|\vec{v}_1\| = 1}}$

$\vec{v}_2 = \begin{bmatrix}0\\2\\1\\2\end{bmatrix}$ $\quad \|\vec{v}_2\| = \sqrt{4+1+4}$ $\underline{\underline{= 3}}$

$\vec{v}_3 = \begin{bmatrix}0\\1/9\\-4/9\\1/9\end{bmatrix}$

$\|\vec{v}_3\| = \frac{1}{9}\cdot\sqrt{1+16+1} = \frac{\sqrt{18}}{9}$

$\underline{\underline{= \frac{\sqrt{2}}{3}}}$

Orthonormal basis: $\quad \vec{u}_i = \frac{1}{\|\vec{v}_i\|}\cdot\vec{v}_i$

$$\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2/3\\1/3\\2/3\end{bmatrix}, \begin{bmatrix}0\\ \frac{1}{3\sqrt{2}}\\ \frac{-2\sqrt{2}}{3}\\ \frac{1}{3\sqrt{2}}\end{bmatrix} \right\}$$

typo

10. By drawing a direction field, sketch two solutions to the ODE

$$\frac{dy}{dt} = y^2(y-2)$$

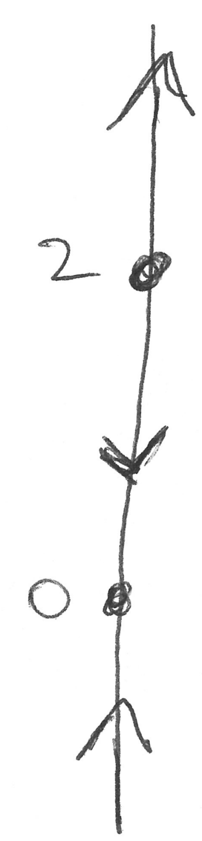
with initial conditions $y(0) = 1$ and $y(0) = 3$.

Indicate clearly the limiting behavior $\lim_{t \to \infty} y(t)$ and $\lim_{t \to -\infty} y(t)$.

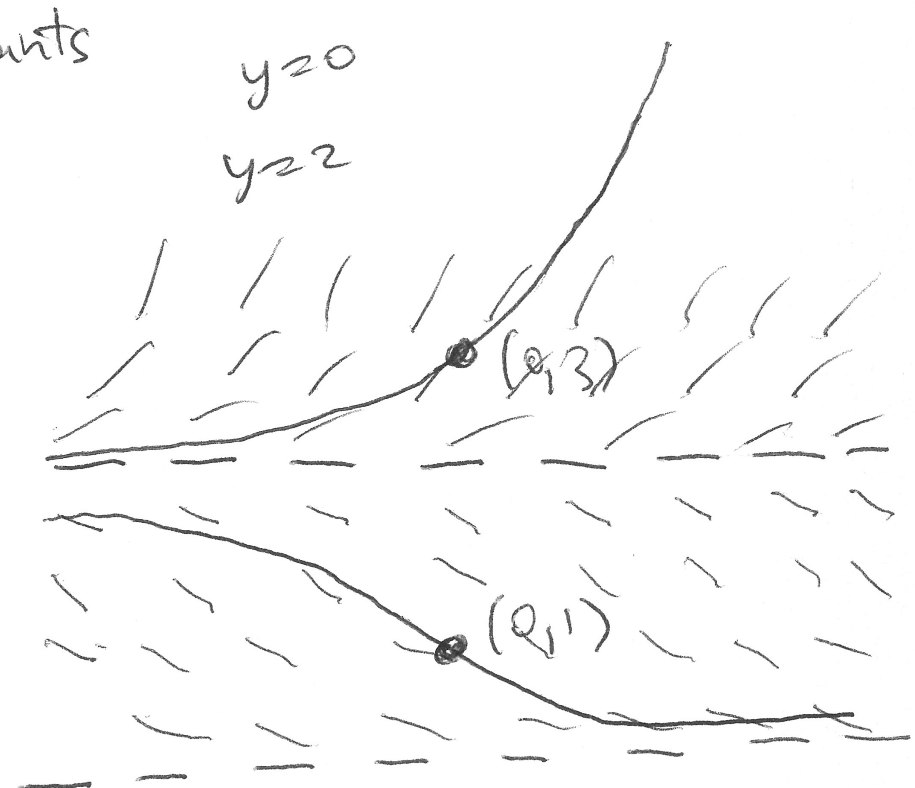
Critical points

$y = 0$

$y = 2$



phase line



11. Find the function $y(t)$, for $t > 0$, which solves the initial value problem

$$t\frac{dy}{dt} + 4y = \frac{e^{-t}}{t^2} \quad , \quad y(1) = 0$$

$$\frac{dy}{dt} + \underbrace{\frac{4}{t}}_{p(t)} y = \underbrace{\frac{e^{-t}}{t^3}}_{g(t)}$$

$$\mu(t) = e^{\int p(t)\,dt} = e^{4\ln t} = t^4 \quad \text{integrating factor}$$

$$y(t) = \frac{\int \mu(t)\cdot g(t)\,dt}{\mu(t)}$$

$$\int t^4 \cdot \frac{e^{-t}}{t^3}\,dt = \int t e^{-t}\,dt = t\cdot(-e^{-t}) - \int(-e^{-t})\,dt$$

$$= -te^{-t} - e^{-t} + C$$

So $$y(t) = \frac{-e^{-t}(t+1) + C}{t^4}$$

$$y(1) = 0 \Rightarrow \boxed{C = 2e^{-1}}$$

$$\boxed{y(t) = \frac{2e^{-1} - e^{-t}(t+1)}{t^4}}$$

12. Consider the differential equation

$$2y\frac{dy}{dx} = -e^x$$

(a) Find the general solution.
(b) Find the solution with $y(0) = 1$.
(c) What is the largest interval in which the solution in part (b) is defined?

(a) $\int 2y\,dy = \int -e^x\,dx$

$y^2 = -e^x + C$

$y = \pm\sqrt{-e^x + C}$

(b) $y(0) = 1 \Rightarrow$ positive sign

$1 = \sqrt{-e^0 + C} \Rightarrow \underline{C = 2}$

$y = \sqrt{2 - e^x}$

(c) Need $2 - e^x \geq 0$ so

$e^x \leq 2$

$\Leftrightarrow \boxed{x \leq \ln 2}$

$\boxed{I = (-\infty, \ln 2]}$