

Math 20580
Midterm 3
April 16, 2015

Name: Solutions
Instructor: _____
Section: _____

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an \times through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

1. a b c d e

2. a b c d e

3. a b c d e

4. a b c d e

5. a b c d e

6. a b c d e

7. a b c d e

8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. The closest point to $\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ in the subspace of \mathbb{R}^3 spanned by $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ is

- (a) $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 1/2 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ (e) $\begin{bmatrix} 9 \\ 12 \\ 0 \end{bmatrix}$

This is $\text{proj}_{\text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}\right\}} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$.

As the spanning vectors are orthogonal, this is

$$\frac{\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}}{24} \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

2. Consider the vector $\vec{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and the line $W = \text{Span}\{\vec{w}\}$ spanned by \vec{w} . $A\vec{x}$ computes the projection of a vector \vec{x} onto W if

- (a) $A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$ (b) $A = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$
 (d) $A = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$ (e) $A = \begin{bmatrix} 1/10 & -3 \\ -3/10 & 9 \end{bmatrix}$

$$\text{proj}_W \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \text{proj}_W \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-3}{10} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

3. The matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}$ factors as $A = QR$ where $Q = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and

(a) $R = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$ (b) $R = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ (c) $R = \begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix}$

(d) $R = \begin{bmatrix} 1/2 & -3/10 \\ 0 & 1/5 \end{bmatrix}$ (e) $R = \begin{bmatrix} 5 & 0 \\ -3 & 2 \end{bmatrix}$

$$R = Q^T A = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$$

4. Let A be an $m \times n$ matrix and let \vec{b} be a vector in $\text{Col}(A)$. Which of the following statements may be false?

- (a) There exists a vector \vec{x} in \mathbb{R}^n with $A\vec{x} = \vec{b}$.
- (b) If A has linearly independent columns then $\det(A^T A) \neq 0$.
- (c) \vec{b} is perpendicular to every vector in the null space of A^T .
- (d) If $A^T \vec{x} = A^T \vec{b}$ then \vec{x} is contained in $\text{Col}(A)$.
- (e) If $m > n$ then $\det(AA^T) = 0$.

(a) is true as b lies in $\text{Col}(A) = \text{range}(T_A)$

(b) is true since the normal equations have a unique solution

(c) true as $\text{Col}(A) = \text{Nul}(A^T)^\perp$

(d) May be false, eqn. says $x-b$ lies in $\text{Nul}(A^T) = \text{Col}(A)^\perp$

(e) true as $\text{Nul}(A^T) \neq \{0\}$.

5. Find the solution to the initial value problem

$$t \frac{dy}{dt} + y = t \sin t, \quad y(\pi) = 1.$$

(a) $y = -\sin t - \pi \frac{\cos t}{t}$

(b) $y = -\cos t + \pi \frac{\sin t}{t}$

(c) $y = -\cos t + t \sin t$

(d) $y = -\cos t + \frac{\sin t}{t}$

(e) $y = -\cos t + 2 \frac{\sin t}{t^2}$

ODE is linear, in standard form it's $y' + (\frac{1}{t})y = \sin t$.

The integrating factor $\mu = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$,

so a general solution is $y = \frac{1}{t} \left(\int t \sin t + C \right)$

$$\Rightarrow y = -\cos t + \frac{\sin t}{t} + \frac{C}{t} \quad y(\pi) = 1 \Rightarrow C = 0.$$

6. Consider the initial value problem

$$\frac{dy}{dt} = y^2 - y - 2, \quad y(0) = 0.$$

Which of the following describes the nature of the solution?

(a) $\lim_{t \rightarrow -\infty} y(t) = -1$; $\lim_{t \rightarrow \infty} y(t) = 2$; inflection point at $y = 1/2$

(b) $\lim_{t \rightarrow -\infty} y(t) = 2$; $\lim_{t \rightarrow \infty} y(t) = -1$; inflection point at $y = 1$

(c) $\lim_{t \rightarrow -\infty} y(t) = -2$; $\lim_{t \rightarrow \infty} y(t) = 1$; inflection point at $y = 0$

(d) $\lim_{t \rightarrow -\infty} y(t) = -1$; $\lim_{t \rightarrow \infty} y(t) = 2$; inflection point at $y = 1$

(e) $\lim_{t \rightarrow -\infty} y(t) = 2$; $\lim_{t \rightarrow \infty} y(t) = -1$; inflection point at $y = 1/2$

The phase line is

So $\lim_{t \rightarrow \infty} y = -1$, $\lim_{t \rightarrow -\infty} y = 2$ (since start at $y(0) = 0$).

Inflection pt. is when $(y^2 - y - 2)' = 0$ or $y = 1/2$

7. Let $y(t)$ be the unique solution of the initial value problem

$$\cos t \frac{dy}{dt} + t^2 y = \frac{1}{t-1} \quad y(0) = C$$

What is the largest interval on which you can guarantee a solution y exists?

- (a) $t > 0$ (b) $\frac{-\pi}{2} < t < 1$ (c) $t < 1$ (d) $-1 < t < 1$ (e) $\frac{-\pi}{2} < t < \frac{\pi}{2}$

In standard form, this is

$$y' + \underbrace{\left(\frac{t^2}{\cos t}\right)}_{p(t)} y = \underbrace{\frac{1}{\cos t (t-1)}}_{g(t)}$$

$p(t)$ singular if $t = \frac{\pi}{2} + n\pi$

$g(t)$ singular if $t = \frac{\pi}{2} + n\pi$ or $t = 1$.

8. Find an implicit solution to the initial value problem

$$y + (x + e^y) \frac{dy}{dx} = 0, \quad y(1) = 0.$$

- (a) $x + e^y = 2$ (b) $y + xe^y = 1$ (c) $xy + e^y = 2$
(d) $xy + xe^y = 1$ (e) $xy + e^y = 1$

Set $M = y$, $N = x + e^y$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 1$ so ODE is exact.

$$\frac{\partial \Psi}{\partial x} = y \Rightarrow \Psi = xy + h(y)$$

$$\frac{\partial \Psi}{\partial y} = x + e^y \Rightarrow h'(y) = e^y \Rightarrow h = e^y + C$$

So general sol. is $xy + e^y = C$, $y(1) = 0 \Rightarrow C = 1$

Part II: Partial credit questions (11 points each). Show your work.

9. Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 6 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthogonal basis \mathcal{B} for W .

$$\text{Set } \gamma_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} - \frac{v_2 \cdot \gamma_1}{\gamma_1 \cdot \gamma_1} \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 6 \\ 0 \\ 3 \\ 1 \end{pmatrix} - \frac{v_3 \cdot \gamma_1}{\gamma_1 \cdot \gamma_1} \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix} - \frac{v_3 \cdot \gamma_2}{\gamma_2 \cdot \gamma_2} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 3 \\ 1 \end{pmatrix} - \frac{19}{19} \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix} - \frac{9}{9} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 4 \\ 0 \end{pmatrix}$$

(b) Decide whether the vector $\vec{x} = \begin{bmatrix} 9 \\ 9 \\ 0 \\ 0 \end{bmatrix}$ is contained in W , and if it is then find the

coordinate vector $[\vec{x}]_{\mathcal{B}}$ of \vec{x} with respect to the basis \mathcal{B} found in part (a).

$$\text{proj}_W \begin{pmatrix} 9 \\ 9 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \frac{36}{9} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \frac{18}{18} \begin{pmatrix} 1 \\ 1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 0 \\ 0 \end{pmatrix}$$

so $\begin{pmatrix} 9 \\ 9 \\ 0 \\ 0 \end{pmatrix}$ lies in the subspace and its coordinates are $\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$.

10. Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 4 \end{bmatrix}$.

(a) Find the least square solution to the equation $A\vec{x} = \vec{b}$.

$$A^T A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ -3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 10 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

So least square soln. satisfies $\begin{pmatrix} 6 & -3 \\ -3 & 10 \end{pmatrix} x = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

$$\text{or } x = \frac{1}{51} \begin{pmatrix} 10 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 9 \\ 4 \end{pmatrix} = \frac{1}{51} \begin{pmatrix} 102 \\ 51 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(b) Find the vector in the image of A which is closest to \vec{b} .

$$Ax = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 4 \\ 2 \end{pmatrix}$$

11. At time $t = 0$ a tank contains 50 gallons of water mixed with 10 kg. of salt. Over time well-mixed salt water drains from the tank at a rate of 2 gallons per day and is replenished with pure water which enters the tank at a rate of 3 gallons per day.

(a) Set up an initial value problem for the mass Q of salt remaining in the tank at time t .

$$\begin{aligned}\frac{dQ}{dt} &= (\text{rate salt enters}) - (\text{rate salt exits}) \\ &= 0 - 2 \cdot \frac{Q}{50+t} \quad \text{since pure water enters and the volume is } 50+t. \\ Q(0) &= 10\end{aligned}$$

(b) Solve the initial value problem to derive an explicit formula for Q .

The equation is separable, so

$$\int \frac{dQ}{Q} = -2 \int \frac{dt}{50+t}$$

$$\text{so } \ln Q = -2 \ln(50+t) + C \quad \text{or } Q = \frac{A}{(50+t)^2}$$

$$Q(0) = 10 \Rightarrow 10 = \frac{A}{50^2} \Rightarrow A = 25000$$

$$\text{so } Q = \frac{25000}{(50+t)^2}$$

12. Find an implicit solution to the initial value problem

$$y^2 + (xy + 2)\frac{dy}{dx} = 0, \quad y(0) = 1.$$

$$\text{Set } M = y^2, \quad N = xy + 2$$

$$M_y = 2y, \quad N_x = y, \quad \text{so eqn. not exact.}$$

$$\text{But } \frac{N_x - M_y}{M} = \frac{-1}{y}, \quad \text{so there will}$$

be an integrating factor depending on y .

$$\text{This satisfies } \mu' = \frac{-1}{y} \mu$$

$$\text{or } \ln \mu = -\ln y + C. \quad \text{Set } \mu = \frac{1}{y}.$$

$$\text{Multiplying gives } y + \left(\cancel{x} + \frac{2}{y}\right) y' = 0$$

which is exact.

$$\text{The potential has } \Psi_x = y \Rightarrow \Psi = xy + h(y)$$

$$\Psi_y = x + h'(y) \Rightarrow \text{set } h = 2 \ln y.$$

$$\text{General soln. is } \Psi = xy + 2 \ln y = C. \quad \text{Initial condition gives } C = 0.$$