1.(6pts) Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \right\}$. (Note that the two given vectors spanning W are *not* orthogonal). The projection of $\begin{bmatrix} 3 \\ -8 \\ 4 \end{bmatrix}$ on W is $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. What is the value of a?

(a) 3 (b) 4 (c) 2 (d) -8 (e) 0

2.(6pts) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$. Then b_{32} is equal to:

(a) $-\frac{1}{6}$ (b) $-\frac{1}{3}$ (c) 0 (d) $\frac{1}{6}$ (e) $\frac{1}{3}$

3.(6pts) Let y be the solution to the initial value problem $y'' - 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 4$. Which of the following is equal to $y(2)$?

(a) $3e^3$ (b) $6e^3$ (c) $3e^6$ (d) e^6 (e) $4e^6$

4.(6pts) Let $\mathcal{B}$ be the basis of $\mathbb{R}^2$ given by the vectors $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$ and let x be the vector $x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. The coordinate vector $[x]_{\mathcal{B}}$ of x with respect to $\mathcal{B}$ is $[x]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$. Which of the following is the value of a?

(a) 4 (b) 2 (c) -6 (d) -2 (e) -3

5.(6pts) Let $y(t)$ be the solution to the logistic equation $y' = y(2-y)$ with initial value $y(0) = 1$. What is the value of $\lim_{t\to\infty} y(t)$?

(a) 0 (b) 1 (c) $\dfrac{2e^2}{3-e^2}$

(d) 2 (e) $+\infty$

6.(6pts)Suppose that $y_1(t) = e^t$ and $y_2(t) = t$ are two fundamental solutions to the homogeneous equation

$$(1-t)y'' + ty' - y = 0,$$

and that $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ is a particular solution to the equation

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}.$$

Then $u_2(t)$ could be equal to:

(a) $-2e^{-t}$ (b) $3e^{-t}$ (c) $2te^{-t}$

(d) $-2te^{-t}$ (e) $2te^{-t} - 4e^{-t}$

7.(6pts) Find the least squares solution of the equation $\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ (b) $\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$ (c) $\begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

8.(6pts) Let

$$A = \begin{bmatrix} 2 & 4 & -6 & 5 & -1 \\ 3 & 7 & -11 & 8 & -3 \\ 3 & 6 & -9 & 7 & -2 \end{bmatrix}.$$

Which of the following sets of vectors is a basis of the null space of A?

(a) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

9.(6pts) Which of the following is the general solution for the differential equation

$$(x+2y)\mathrm{d}x - x\mathrm{d}y = 0\,?$$

(First find an integrating factor of the form $\mu(x)$).

(a) $\dfrac{x^2}{2} + 2xy - xy = C$ (b) $y - x = Cx^2$ (c) $x(x+2y) = C$

(d) $y + x = Cx^2$ (e) $xy + y^2 - \dfrac{x^2}{2} = C$

10.(6pts) Which one of the following is a set of orthogonal vectors obtained by applying the Gram-Schmidt process to $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$

11.(6pts) Which of the following is equal to the matrix R in the QR-decomposition $A = QR$ of the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$?

(a) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$

(d) $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$ (e) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

12.(6pts) Consider the initial value problem

$$(4 - x^2)y'' - 2xy' + (\sqrt{1 + \ln|x|})y = 0, \quad y(1) = 0.$$

Which of the following is the largest interval on which a solution y of this initial value problem is guaranteed to exist?

(a) $\frac{1}{e} < x < 2$ (b) $0 < x < 2$ (c) $\frac{1}{e} < x < \infty$

(d) $2 < x < e$ (e) $0 < x < e$

13.(6pts) Suppose that the differential equation $ty'' + 2y' + te^t y = 0$ has $y_1(t)$ and $y_2(t)$ as a fundamental set of solutions. Let $w(t) = W(y_1, y_2)(t)$ be the Wronskian of y_1 and y_2. If $w(1) = 4$, which of the following is equal to $w(2)$?

(a) 4 (b) 1 (c) 2 (d) -2 (e) -4

14.(6pts) Let $y(t)$ be the solution of the initial value problem

$$y'' - 5y' - 6y = 0,\ y(0) = 4,\ y'(0) = \alpha.$$

Suppose $\lim_{t \to +\infty} y(t) = 0$. What is the value of α?

(a) -4 (b) 0 (c) 24 (d) 10 (e) 3

15.(6pts) Use the method of reduction of order to find a second solution $y_2(t)$ of the differential equation $y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 0$ for $t > 0$ given that $y_1(t) = t^2$ is a solution.

(a) $y = \ln t$ (b) $y = t \ln t$ (c) $y = t$

(d) $y = \frac{1}{t}$ (e) $y = t^2 \ln t$

16.(6pts)Find the values of a and b such that the following differential equation is exact:

$$8xy + ay^2 \cos x + (bx^2 + 4y \sin x)y' = 0.$$

(a) $a = 2, b = 1$ (b) $a = 2, b = 4$ (c) $a = 4, b = 2$

(d) $a = 2, b = 8$ (e) $a = 1, b = 4$

17.(6pts)Find the solution of the initial value problem

$$y' = \frac{t-1}{y-1}, \quad y(0) = -1.$$

(a) $y(t) = -1 - \sqrt{t^2 - 2t + 4}$

(b) $y(t) = -1 + \sqrt{t^2 - 2t}$

(c) $y(t) = 1 - \sqrt{t^2 - 2t + 4}$

(d) $y(t) = -1 - \sqrt{t^2 - 2t}$

(e) $y(t) = 1 + \sqrt{t^2 - 2t + 4}$

18.(6pts) Suppose that the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & -4 \\ 2 & -1 & 3 \end{bmatrix}$ is written in the form $A = PDP^{-1}$ where P and D are 3×3 matrices such that P is invertible and D is diagonal. Which of the following numbers is equal to the product of the three numbers appearing on the main diagonal of D?

(a) 3 (b) 4 (c) 6 (d) 2 (e) 5

19. (6pts) Let $\mathcal{P}_2$ be the vector space of polynomials of degree 2 or less with its standard basis $\mathcal{B} = \{1, t, t^2\}$. Let $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation given by $T(p) = (t+2)p'(t)$ where the derivative of a polynomial $p = p(t)$ is denoted as $p'(t)$. Which of the following is the matrix of T with respect to the basis $\mathcal{B}$?

(a) $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

20. (6pts) No springs on the final. A mass weighing 50kg stretches a vertical spring 5m. If there is no damping, which of the following equations describe the motion of the spring, with respect to the equilibrium position $y = 0$? (Use $g = 9.8\text{ms}^{-2}$ for the acceleration due to gravity.)

(a) $50y'' + 5y = 0$ (b) $50y' + 98y = 0$ (c) $50y'' + 98y = 0$

(d) $5y'' + 50y = 0$ (e) $50y'' + 10y = 0$

21.(6pts) Which of the following numbers can *not* be the dimension of the null space of a 3×6 matrix?

(a) 5 (b) 6 (c) 2 (d) 4 (e) 3

22.(6pts) Find the determinant of the matrix below:

$$\begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & -4 \\ 3 & -5 & 6 \end{bmatrix}.$$

(a) -2 (b) 2 (c) 1 (d) 0 (e) -1

23.(6pts) Find the form of a particular solution to

$$y'' + 4y = t^2 \cos 2t$$

using the method of undetermined coefficients.

(a) $y = (A_0 + A_1 t + A_2 t^2) \cos 2t$

(b) $y = (A_0 t + A_1 t^2 + A_2 t^3) \cos 2t$

(c) $y = (A_0 + A_1 t + A_2 t^2) \cos 2t + (B_0 + B_1 t + B_2 t^2) \sin 2t$

(d) $y = (A_0 t + A_1 t^2 + A_2 t^3) \cos 2t + (B_0 t + B_1 t^2 + B_2 t^3) \sin 2t$

(e) $y = (A_0 t^2 + A_1 t^3 + A_2 t^4) \cos 2t + (B_0 t^2 + B_1 t^3 + B_2 t^4) \sin 2t$

24.(6pts) Consider the system of linear equations

$$\begin{cases} (3-s)\,x & - & 4\,y & = & 2 \\ -2\,x & + & (3+s)\,y & = & 4 \end{cases}$$

for x and y, where $s \neq \pm 1$. What is the value of y in the solution?

(a) $y = \dfrac{s+11}{2(1-s^2)}$ (b) $y = \dfrac{2s+22}{1-s^2}$ (c) $y = \dfrac{-s-11}{2(1-s^2)}$

(d) $y = \dfrac{16-4s}{1-s^2}$ (e) $y = \dfrac{4s-16}{1-s^2}$

Solutions

1. Using Gram-Schmidt, we find an orthogonal basis for $W = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\-3\end{bmatrix}\right\}$.

Let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}2\\0\\-3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}5\\0\\-5\end{bmatrix}$. After scaling $\mathbf{v}_2$, we have $W = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}5\\0\\-5\end{bmatrix}\right\}$. The projection of $\begin{bmatrix}3\\-8\\4\end{bmatrix}$ on W is

$$\hat{\mathbf{y}} = \frac{7}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} + \frac{-5}{50}\begin{bmatrix}5\\0\\-5\end{bmatrix} = \begin{bmatrix}3\\0\\4\end{bmatrix}$$

so $a = 3$.

2. We find b_{32} with the inverse formula. Note $\det A = 6$ and $C_{23} = (-1)^5 \begin{vmatrix}1 & 1\\1 & 0\end{vmatrix} = 1$, so $b_{32} = \frac{1}{\det A} C_{23} = \frac{1}{6}$.

3. The characteristic equation is $r^2 - 6r + 9 = 0$ i.e. $(r-3)^2 = 0$. It has a double root $r = 3, 3$, so the general solution is $y = c_1 e^{3t} + c_2 t e^{3t}$. So $y' = 3c_1 e^{3t} + c_2(3te^{3t} + e^{3t})$. The initial conditions give $c_1 + c_2 = 1$ and $3c_1 + c_2 = 4$, from which $c_1 = c_2 = 1$. So $y(t) = e^{3t} + te^{3t}$ and $y(2) = 3e^6$.

4. To find a and b, we row reduce the augmented matrix:

$$\begin{bmatrix}1 & -2 & 2\\-2 & 2 & 4\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 2\\0 & -2 & 8\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -6\\0 & 1 & -4\end{bmatrix}.$$

Thus $[x]_{\mathcal{B}} = \begin{bmatrix}-6\\-4\end{bmatrix}$.

5. This is an autonomous equation $y' = f(y)$ with equilibrium solutions $y = 0$ and $y = 2$ given by the roots of $f(y) = 0$. For $0 < y < 2$, the solution is increasing since $f(y) > 0$, and for $y > 2$ the solution is decreasing since $f(y) < 0$. The solution starting with $y(0) = 1$ will increase towards the equilibrium solution $y = 2$, so $\lim_{t\to\infty} y(t) = 2$. (This may also be checked by explicitly solving the equation; it is separable).

6. In standard form the equation is $y'' + p(y)y' + q(t)y = g(t)$, which is

$$y'' + \frac{t}{1-t}y' + \frac{-1}{1-t}y = 2(1-t)e^{-t}.$$

so $g(t) = 2(t-1)e^{-t}$. The Wronskian is $W(y_1, y_2)(t) = \begin{vmatrix}e^t & t\\e^t & 1\end{vmatrix} = e^t(1-t)$ and

$$u_2'(t) = \frac{y_1(t)g(t)}{W(y_1,y_2)(t)} = \frac{e^t \cdot 2(t-1)e^{-t}}{e^t(t-1)} = 2e^{-t}$$

so $u_2(t) = \int 2e^{-t}\mathrm{d}t = -2e^{-t} + C$. Choose $C = 0$ to get the solution in the answers.

7. We have $A^TA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ and $A^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note $(A^TA)^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$ and so the least squares solution is

$$(A^TA)^{-1}A^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

8. We row reduce:

$$\begin{bmatrix} 2 & 4 & -6 & 5 & -1 \\ 3 & 7 & -11 & 8 & -3 \\ 3 & 6 & -9 & 7 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 5/2 & -1/2 \\ 0 & 1 & -2 & 1/2 & -3/2 \\ 0 & 0 & 0 & -1/2 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 5/2 & -1/2 \\ 0 & 1 & -2 & 1/2 & -3/2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 & -3 \\ 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The pivot columns are 1, 2, and 4 so x_3 and x_5 are free variables. Writing the homogeneous equations from the row reduced matrix we have

$$\begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= 2x_3 + 2x_5 \\ x_3 &= -x_5 \end{aligned}$$

Setting $x_3 = 1$ and $x_5 = 0$ gives the vector $[-1, 2, 1, 0, 0]^T$ and setting $x_3 = 0$ and $x_5 = 1$ gives the vector $[-1, 2, 0, -1, 1]^T$.

9. The equation is of the form $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$. The form suggests looking for an integrating factor to make it exact. One has $M = x + 2y$, $N = -x$, $L = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3$. Since $L/N = -3/x$,we try an integrating factor $\mu(x)$ with $\mu' = (-3/x)\mu$. This gives $\ln \mu = -3 \ln x$, $\mu = x^{-3}$. The equation becomes $x^{-3}(x + 2y)\mathrm{d}x - x^{-3}x\mathrm{d}y = 0$, which is exact. We seek $P = P(x, y)$ with $P_x = x^{-2} + 2yx^{-3}$, $P_y = -x^{-2}$. The first condition gives $P(x, y) = -x^{-1} - yx^{-2} + h(y)$. Substituting in the second gives $h' = 0$, so we may take $h = 0$ and $P(x, y) = -x^{-1} - yx^{-2}$. The solution is $P(x, y) = c$ i.e. $-1/x - y/x^2 = c$ or $x + y = -cx^2$. This is $x + y = Cx^2$ with $C = -c$.

(The equation may also be solved as a homogeneous equation. It is $y' = (1 + 2y/x)$ where $1 + 2y/x$ is a function of y/x. So we substitute $y/x = v$, $y = vx$, $y' = v' + v$. The equation becomes $v' + v = 1 + 2v$ or $v' = 1 + v$. This is separable. Integrating, $\int \frac{\mathrm{d}v}{1+v} = \int \mathrm{d}x$ i.e. $\ln|v+1| = \ln|x| + c$. Exponentiating, $|v+1| = e^c|x|$ and $v+1 = Cx$ where $C = \pm e^c$. So $y/x + 1 = Cx$ and $y + x = Cx^2$.)

10. Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$. The Gram-Schmidt process yields $\mathbf{v}_1 = \mathbf{x}_1$ and

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{7}{14}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}.$$

Scaling $\mathbf{v}_2$ by 2 gives us the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} \right\}$

11. An orthogonal basis for the column space of A is $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We normalize the two vectors to get $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ We find R by computing

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}.$$

12. This is a linear equation. In standard form $y'' + p(x)y' + q(x)y = g(t)$, it is

$$y'' + \frac{-2x}{4-x^2}y' + \frac{\sqrt{1+\ln|x|}}{4-x^2}y = 0.$$

The solution exists in the largest open interval containing $x = 1$ on which p, q and g are defined and continuous. Note p is undefined at $x = \pm 2$ and q is undefined where $x = 0$ or $\ln|x| < -1$ i.e $|x| < 1/e$. The largest interval on which the solution exists is $1/e < x < 2$.

13. The equation is $y'' + p(t)y' + q(t)y = 0$ where $p(t) = 2/t$ and $q(t) = e^t$ The Wronskian $w(t)$ satisfies the ODE $w'(t) = -p(t)w(t)$. It is of the form $w(t) = Ce^{-\int p(t)\mathrm{d}t} = Ce^{-\int 2t^{-1}\mathrm{d}t} = Ce^{-2\ln t} = C/t^2$ for some constant C. Since $w(1) = 4$, $C = 4$. So $w(2) = 1$.

14. The characteristic equation is $r^2 - 5r - 6 = 0$ i.e. $(r-6)(r+1) = 0$ with roots $r = 6, -1$. The general solution is $y(t) = c_1e^{6t} + c_2e^{-t}$, so$y'(t) = 6c_1e^{6t} - c_2e^{-t}$. From the initial values, $4 = c_1 + c_2$ and $\alpha = 6c_1 - c_2$. Solving, $c_1 = (4+\alpha)/7$ and $c_2 = (24-\alpha)/7$. For $\lim_{t\to+\infty} y(t) = 0$, we need $c_1 = 0$ and $\alpha = -4$.

15. The equation is $y'' + p(t)y' + q(t)y = 0$ with solution $y_1(t) = t^2$. To find another solution by reduction of order, substitute $y = y_1v$ to get $y_1v'' + (py_1 + 2y_1')v' = 0$ i.e. $t^2v'' + (-3t+4t)v' = 0$ i.e. $v'' + \frac{1}{t}v' = 0$. This is linear with integrating factor $\mu(t) = e^{\int t^{-1}\mathrm{d}t} = e^{\ln t} = t$ since $t > 0$. We get $(tv')' = 0$, $tv' = C$, $v' = c/t$ and $v = C\ln t + D$. So $y = t^2\ln t$ is another solution.

16. The equation $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Here, $M = 8xy + ay^2\cos x$ and $N = bx^2 + 4y\sin x$, so $\frac{\partial M}{\partial y} = 8x + 2ay\cos x$ and $\frac{\partial N}{\partial x} = 2bx + 4y\cos x$. These are equal if $b = 4$ and $a = 2$.

17. The equation is separable: separating and integrating gives $(y-1)\mathrm{d}y = (t-1)\mathrm{d}t$ and $y^2/2 - y = t^2/2 - t + C$. So $y^2 - 2y + 1 = t^2 - 2t + (1+2c)$, $y - 1 = \pm\sqrt{t^2 - 2t + (1+2c)}$ and $y = 1 \pm \sqrt{t^2 - 2t + (1+2c)}$. The initial value shows $-1 = 1 \pm \sqrt{1+2c}$ so the sign is "$-$", $\sqrt{1+2c} = 2$, $1 + 2c = 4$ and $y(t) = 1 - \sqrt{t^2 - 2t + 4}$.

18. The determinant of A is equal to the determinant of D. Since D is a diagonal matrix, its determinant is the product of its diagonal entries. The determinant of A is $3(2 \cdot 3 - (-1)(-4)) = 6$.

19. We compute

$$\begin{aligned} T(1) &= 0 \\ T(t) &= t + 2 \\ T(t^2) &= (t+2)(2t) = 2t^2 + 4t \end{aligned}$$

It follows the matrix of T with respect to $\mathcal{B}$ is $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.

20. At equilibrium, the upward spring force $5k$ (where k is the spring constant) is equal in magnitude to the weight $9.8 \cdot m$ where $m = 50$ is the mass, giving $k = 9.8 \cdot 50/5 = 98$. After displacement of y from equilibrium, the net force is $-ky = my''$ by Newton's second law, giving the equation $50y'' + 98y = 0$.

21. By the Rank Theorem, if A is a 3×6 matrix, the dimension of the null space is $6 -$ rank A. Since A has 3 rows, the rank is at most 3. Thus the dimension of the null space is at least 3.

22. Expanding along the first row, we have the determinant is $1(18-20) + 1(-12+12)) + 1(10-9) = -2 + 1 = -1$.

23. The form is $y = t^s[(A_0 + A_1 t + A_2 t^2)\cos 2t + (B_0 + B_1 t + B_2 t^2)\sin 2t]$ where s in $0, 1, 2$ is the smallest number so none of the terms involved is a solution of the corresponding homogeneous equation. The homogeneous equation has characteristic equation $r^2 + 4 = 0$ with roots $r = \pm 2i$, and so its fundamental solutions are $\sin 2t$ and $\cos 2t$. We must take $s \geq 1$ so $t^s \cos 2t$ and $t^s \sin 2t$ are not solutions of the homogeneous equation, and $s = 1$ is the smallest choice.

24. We use Cramer's rule. Let $A = \begin{bmatrix} 3-s & -4 \\ -2 & 3+s \end{bmatrix}$, $A_2 = \begin{bmatrix} 3-s & 2 \\ -2 & 4 \end{bmatrix}$. Then $\det A = (3-s)(3+s) + 8 = 9 - s^2 + 8 = 1 - s^2$ and $\det A_2 = 4(3-s) + 4 = 16 - 4s$. By Cramer's rule, we have

$$y = \frac{\det A_2}{\det A} = \frac{16 - 4s}{1 - s^2}.$$